11th International Conference on

Graph Theory and Algebraic Combinatorics

March 4-5, 2021

Department of Mathematics,
Urmia University, Urmia, Iran

Virtual Conference

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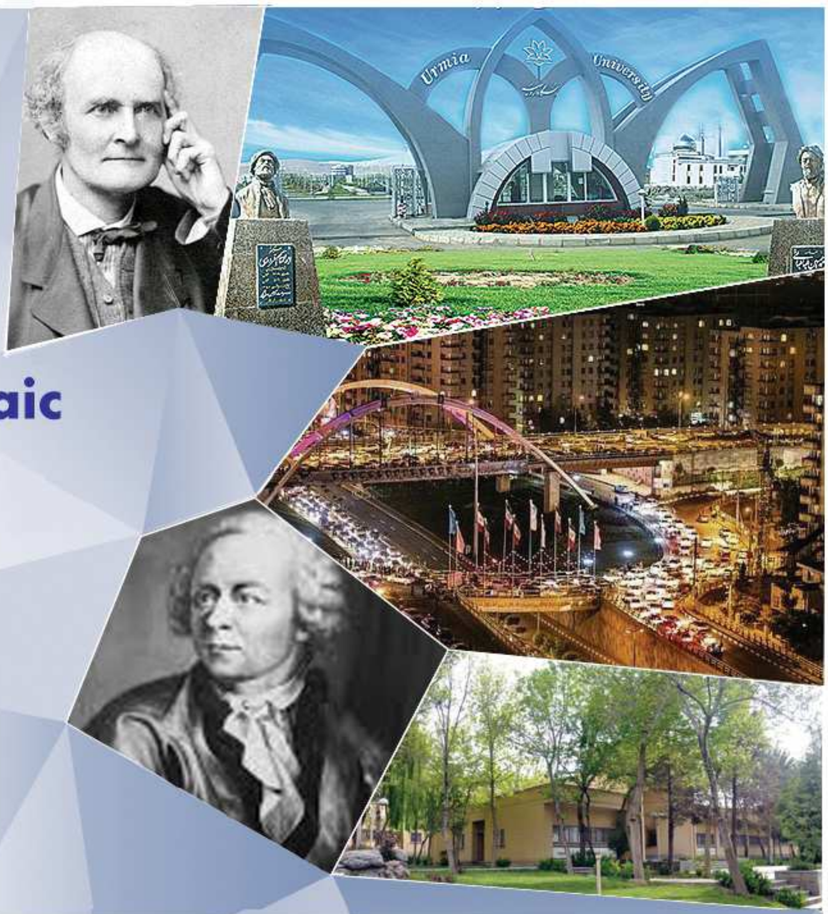

Book of Abstracts

Topics:

- Graph Theory
- Algebraic Graph Theory
- Combinatorics
- Algebraic Combinatorics
- Matroid Theory

Abstracts

11th International Conference on Graph Theory and Algebraic Combinatorics

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Edited By:

Dr. Mohsen Ghasemi

Dr. Vahid Ghorbani

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Preface

Welcome

On behalf of the organizing committee of the "11^{th} International conference on graph theory and algebraic combinatorics", I am pleased to welcome you to the conference.

This conference has come a long way from its first meeting in 2008. The growth of the conference started, from about 50 participants at the first meeting to more than 220 participants at the present conference. There are 14 keynote speakers and 40 contributed talks. The main themes of the conference are: Graph Theory, Algebraic graph theory, Combinatorics, Algebraic combinatorics, Matroid Theory.

Finally, I would like to thank all of my colleagues, all the authors, the reviewers for their contributions and participants. My special gratitude is going to Vahid Ghorbani, Behzad Asgari, and Farnaz Soleimani.

The conference will not be a success without your expertise and active participation.

Mohsen Ghasemi

Chairman

Scientific Committee of 11GTACC

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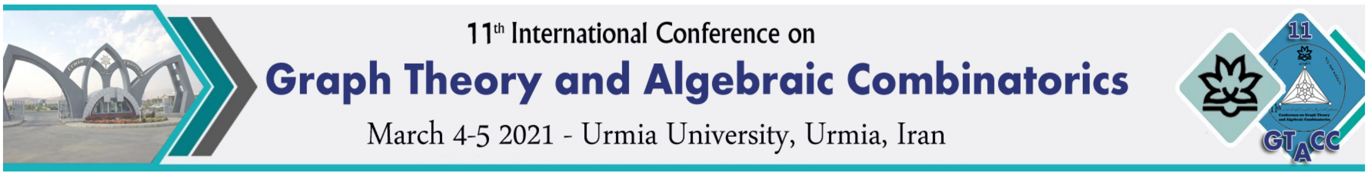
Invited Talks



Unimodality of domination polynomial-A survey

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Abstract

Let G be a graph of order n. A subset S of $V(G)$ is a dominating set of G if every vertex in $V(G)\backslash S$ is adjacent to at least one vertex of S. The domination polynomial of G is the polynomial $D(G,x)=\sum_{i=\gamma(G)}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of size i, and $\gamma(G)$ is the size of a smallest dominating set of G, called the domination number of G. There is a conjecture in [Introduction to domination polynomial of a graph, *Ars Combin.*, **114** (2014) 257-266] which states that the domination polynomial of any graph is unimodal. In this paper, we summarize known results on this subject.

Keywords: domination polynomial, domination number, log-concavity, unimodality.

Mathematics Subject Classification [2010]: Primary: 05C69

1 Introduction

Let $G=(V,E)$ be a simple graph of order $n(G)$ and size $m(G)$. A subset S of $V(G)$ is a dominating set of G if every vertex in $V(G)\backslash S$ is adjacent to at least one vertex of S. The *domination number* of G, $\gamma(G)$ is the minimum size of a dominating set in G. A subset S of $V(G)$ is called an i-*subset* if $|S|=i$. Further, we say S is an *dominating* i-*set* if S is also a dominating set of G. Let $d(G,i)$ be the number of dominating i-set, then

$$D(G,x)=\sum_{i=\gamma(G)}^{n(G)} d(G,i)x^i$$

is known as the domination polynomial of G (see [1, 4]). The concept of domination polynomial was introduced in 2014 [4], but numerous other papers on the polynomial appeared earlier (see, for example [1, 2, 10]). Applications of domination polynomial and dominating sets in solving a variety of network problems can be found in [8, 9]. A number of papers have since been published on the subject A root of $D(G,x)$ is called a domination root of G. We denote the set of all distinct domination roots of G by $Z(D(G,x))$. The corona of two graphs G_1 and G_2, is the graph $G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the i-th vertex of G_1 is adjacent to every vertex in the i-th copy of G_2.

A finite sequence of real numbers $(a_0,a_1,a_2,\ldots,a_n)$ is said to be

1. *unimodal* if $a_0 \le a_1 \le \cdots \le a_{k-1} \le a_k \ge a_{k+1} \ge \cdots \ge a_n$ for some $k \in \{0,1,2,\ldots,n\}$;
2. *logarithmically-concave* (or simply *log-concave*) if the inequality $a_k^2 \ge a_{k-1}a_{k+1}$ is valid for every $k \in \{1,2,\ldots,n-1\}$.

[1] speaker

Hence, a polynomial $\sum_{k=0}^{n} a_k x^k$ is said to be unimodal (or log-concave) if the coefficient sequence $\{a_k\}$ is unimodal (or log-concave). It is well-known that any log-concave polynomial with positive coefficients is also unimodal, and that the sequence of binomial coefficients $\{\binom{n}{k}\}$ is log-concave. The unimodality of some graph polynomials of various families of graphs has been the focus of a large amount of study. Despite the considerable attention paid to unimodality, little is known about the unimodality of the domination polynomial. In this paper, we summarize known results on this subject.

2 Main results

It is conjectured that the domination polynomial of a graph is unimodal (see [4]). This conjecture is still open. Regarding to this conjecture, there is the following result which shows that the coefficients of domination polynomial of graph G of order n are non-decreasing up to $\frac{n}{2}$.

Theorem 2.1. [4] *Let G be a graph of order n. Then for every $0 \leq i < \frac{n}{2}$, we have $d(G,i) \leq d(G,i+1)$.*

We need the following theorems:

Theorem 2.2. [5, 10] *Let $G = (V,E)$ and $H = (W,F)$ be nonempty graphs of order n and m, respectively. Then*

$$D(G \circ H, x) = (x(1+x)^m + D(H,x))^n.$$

Theorem 2.3. [11] *Let $f(x)$ and $g(x)$ be polynomials with positive coefficients. If both $f(x)$ and $g(x)$ are log-concave, then so is their product $f(x)g(x)$.*

The following corollary is an immediate consequence of the above theorem.

Corollary 2.4. *If polynomials $P_i(x)$ for $i = 1,\ldots,k$ with positive coefficients are log-concave, then $\prod_{i=1}^{k} P_i(x)$ is log-concave as well.*

The following theorem gives us a sequence of graphs whose are unimodal:

Theorem 2.5. [3] *Every graph H in the family $\{G \circ K_n, (G \circ K_n) \circ K_n, ((G \circ K_n) \circ K_n) \circ K_n, \cdots\}$ is unimodal.*

Theorem 2.6. [3] *The domination polynomials of F_n are unimodal.*

Recently, Beaton and Brown [7] shown that paths, cycles and complete multipartite graphs are unimodal, and that the domination polynomial of almost every graph is unimodal with mode $\lceil \frac{n}{2} \rceil$. We thought that the domination polynomial of any graph is log-concave and this is true for any graph of order at most 8. The only graph of order 9 which is not log-concave is the graph in Figure 1 ([7]). Note that the domination polynomial of this graph is

$$D(G,x) = x^9 + 9x^8 + 35x^7 + 75x^6 + 89x^5 + 50x^4 + 7x^3 + x^2.$$

which is not log-concave, because $(d(G,3))^2 = 49 < d(G,4)d(G,2) = 50$.

In this talk, after some preliminaries, we summarize the known results on the unimodality of domination polynomial and finally ask some open problems that will solve the conjecture of the unimodality of the domination polynomial.

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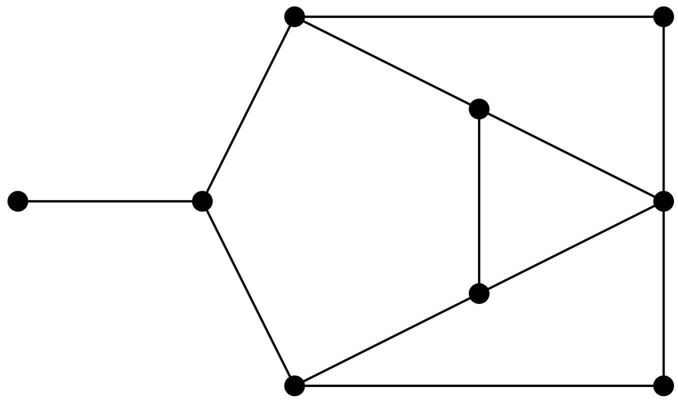

Figure 1: The only graph G of order 9 which is not log-concave.

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Vertex-transitive graphs admitting semiregular automorphisms

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Abstract

In 1981 Marušič asked whether there exists a vertex-transitive digraph without a non-identity automorphism having all of its orbits of the same length. By powering a group element up, the existence of such an automorphism (which is usually called semiregular) is equivalent to the existence of a fixed-point free automorphism of prime order. In 1988, independently, the above problem was again proposed by Jordan. In the 15th British combinatorial conference, in 1995, Klin proposed a more general question in the context of 2-closed groups; Is there a 2-closed transitive permutation group containing no fixed-free element of prime order? Broadly speaking, Klin's question is a graph colored version of the Marušič-Jordan question. A graph admitting a fixed-point free automorphism of prime order is called polycirculant, so it is customary to refer to the conjecture that every 2-closed transitive permutation group admits a fixed-point free element of prime order as the Polycirculant Conjecture. While there has been a lot of works on this conjecture and some of its variants, it is still wide open. In this lecture we talk about recent trends and open problems relating to the conjecture and review some applications of semiregular automorphisms.

Keywords: Fixed-point free automorphism, polycirculant graph, 2-closure.
Mathematics Subject Classification [2010]: Primary: 20B25; Secondary: 05C25, 20B05

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Counting spanning trees

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Abstract

For a connected graph G, let $\tau(G)$ be the number of spanning trees in G. Cayley's formula gives a simple expression for $\tau(K_n)$, where K_n is the complete graph of order n. In this talk, I will introduce some extensions of Cayley's formula, which include formulas for $\tau(L(G))$ and $\tau(M(G))$ for an arbitrary graph G, where $L(G)$ and $M(G)$ are the line graph and the middle graph of G respectively. I will also present some known results and unsolved problems related to $\tau(G)$.

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Old and New Results on the Spectral Radius and Energy of Certain Matrices Associated to a Graph

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Abstract

In the present work, we survey old and new results concerning the spectral radius of special graphs. Besides, we present new results about the largest eigenvalues of these matrices in terms of several graph invariants.

Keywords: graph eigenvalues, spectral radius, Perron-Frobenius theorem.
Mathematics Subject Classification [2010]: 05C50

1 Introduction

Let $G = (V, E)$ be a simple graph with n vertices, m edges with adjacency matrix $A(G)$. The eigenvalues of graph G are the roots of characteristic polynomial $P_G(\lambda) = \det(\lambda I - A(G))$, where I is the identity matrix of order n.

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[1]speaker

s-arc-transitive graphs and digraphs

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Abstract

The study of s-arc-transitive graphs goes back to the seminal work of Tutte in 1947 who showed that in the cubic case $s \leqslant 5$. This has motivated a large body of research on s-arc-transitive graphs and digraphs. I will survey some recent results in the area

Mathematics Subject Classification [2010]: 20B25, 05C25

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[1] speaker

Algorithms on hub allocation problems

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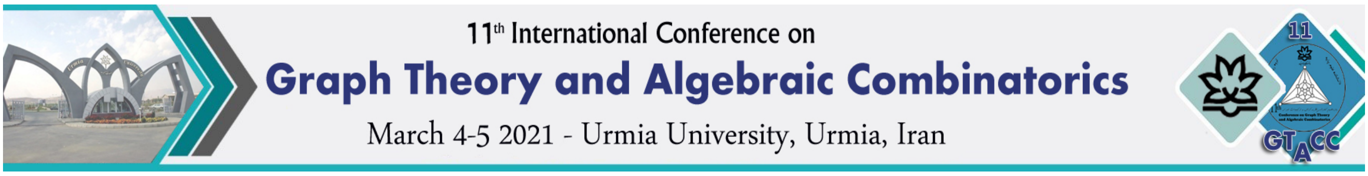
Abstract

Given a metric graph $G = (V, E, w)$, a center , and an integer k, the Star k-Hub Center Problem is to find adepth-2 spanning tree T of G rooted by c such that c has exactly k children and the diameter of T is minimized. Those children of c in T are called hubs. A similar problem called the Single Allocation k-Hub Center Problem is to find a spanning subgraph H* of G such that (i) is a clique of size k in H*; (ii) forms an independent set in H*; (iii) each is adjacent to exactly one vertex in C*; and (iv) the diameter $D(H^*)$ is minimized. The vertices selected in C^* are called hubs and the rest of vertices are called non-hubs. Both Star k-Hub Center Problem and Single Allocation k-Hub Center Problem are NP-hard and have applications in transportation system, telecommunication system, and post mail system. In this talk, we give $\frac{5}{3}$-approximation algorithms for both problems. Moreover, we prove that for any $\varepsilon > 0$, the Star k-Hub Center Problem has no $(1.5 - \varepsilon)$-approximation algorithm unless $P = NP$. Under the assumption $P \neq NP$, for any $\varepsilon > 0$ the Single Allocation k-Hub Center Problem has no $(\frac{4}{3})$-approximation algorithm.

[1]speaker



Parsimonious asymmetrization of graphs

Wilfried Imrich[1]
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Abstract

A coloring of the vertex set of a graph G is *distinguishing* or *asymmetrizing* if the identity automorphism is the only automorphism that preserves it. The minimum number of colors needed for a distinguishing coloring is the distinguishing number $D(G)$. For graphs G with distinguishing number 2 the vertex set $V(G)$ can be partitioned into two sets, each of which is only preserved by the identity automorphism. Such sets are called *asymmetrizing* and the minimum cardinality of such a set is the *2-distinguishing cost* $\rho(G)$ of G. For infinite graphs $\rho(G)$ may be infinite. In that case one looks for sparse asymmetrizing sets and defines a *2-distinguishing density*. Closely related to these parameters is the *motion* $m(G)$ of a graph G. It is the minimum number of vertices moved by each nonidentity automorphism.

The talk treats the relationship between these parameters in general and in more detail for certain classes of graphs. In particular, we consider trees of arbitrary cardinalities, compact trees, graphs of maximum valence 3, and vertex transitive cubic graphs.

On the way we construct cubic vertex transitive graphs with finite motion and positive distinguishing density. The finite versions of such graphs turn out to be Split Praeger–Xu graphs, for which we thus provide another characterization.

---

[1]speaker

Automorphism groups of maps in linear time

Bojan Mohar[1]
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Abstract

In a seminal paper from 1974, Tarjan and Wong (STOC'74) proved that one can decide in linear time whether two planar graphs are isomorphic. Building on Tarjan-Wang's approach and on further work by Kawarabayashi and the speaker (STOC'08), we extend these results and show that the automorphism group of any map (a 2-cell embedded graph) can be found in linear time. The speaker will show the subtleties in the case of spherical maps. This is joint work with Ken-ichi Kawarabayashi, Roman Nedela, and Peter Zeman

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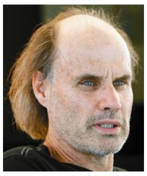
[1]speaker

Vertex-transitive graphs: from semiregular to simplicial automorphisms



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Abstract

When dealing with symmetry of combinatorial objects – or in any other setting for that matter – one will inexorably come across two different kinds of nonidentity automorphisms of these objects: those fixing as large as possible subset of points, on the one hand, and those fixing no points at all on the other. It is the second of these two possibilities that is of interest here. Such automorphisms are called *derangements*. By a theorem of Fein, Kantor and Schacher [3, Theorem 1], every transitive permutation group contains a derangement of prime power order. If we were to replace the requirement that a derangement is of 'prime power order' with that of being of 'prime order' the result would not longer be true. Such groups, although rare, do exist and are called *elusive*, see [2].

Existence of a derangement of prime order in a transitive permutation group is equivalent to the existence of an element with all of its cycles in its cycle decomposition of the same length. Such an element is called *semiregular* and the conjecture regarding existence of such elements in automorphism groups of vertex-transitive (di)graphs (and more general in 2-closed groups) is usually referred to as the *semiregularity problem* (alternatively, the *polycirculant conjecture*), see [1, 5].

In this lecture, I will discuss some recent developments in regards to a special case of the semiregularity problem regarding existence of *simplicial automorphisms*, that is, semiregular automorphisms whose quotient "multigraphs" are simple graphs. When dealing with structural properties of vertex-transitive graphs, reductions via such automorphisms are a viable alternative in absence of normal subgroups. I will give a special emphasis to existence of simplical automorphisms in cubic arc-transitive graphs with a primitive automorphism group [4].

Keywords: At most 5 words or phrases.
Mathematics Subject Classification [2010]: Primary: 22D15, 43A10; Secondary: 43A20, 46H25

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On the smallest signless Laplacian eigenvalue of a graph

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Abstract

The signless Laplacian matrix of a graph is defined as the summation of the adjacency matrix and the diagonal matrix of the vertex degree of the graph. We study the smallest eigenvalue of the signless Laplacian matrix of graphs. We investigate some conjectures related the smallest signless Laplacian eigenvalues of graphs. Using our results, we disprove one of those conjectures.

Keywords: Signless Laplacian Matrix; Smallest signless Laplacian eigenvalue of graphs.
Mathematics Subject Classification [2010]: 05C31, 05C50

1 Introduction

In this paper graphs are simple. In other words, they are finite and undirected, without loops and multiple edges. Let $G = (V, E)$ be a simple graph. The *order* of G denotes the number of vertices of G. For a vertex v of G, the *degree* of v is the number of edges incident with v, and $G \setminus v$ denotes the graph arising from G by deleting the vertex v and all its incident edges. A K-*free* graph is a graph that does not contain K as an induced subgraph. The *complete graph* of order n is denoted by K_n. Let $t \geq 2$ and $n_1, \ldots, n_t$ be some positive integers. By $K_{n_1,\ldots,n_t}$ we mean the *complete multipartite graph* (t-complete multipartite graph) with parts size $n_1, \ldots, n_t$. In particular, K_{n_1,n_2} is called the *complete bipartite* graph. The Turán graph $T(n, t)$ (where $n \geq t \geq 2$ are integer) is the complete multipartite graph formed by partitioning a set of n vertices into t subsets, with sizes as equal as possible, and connecting two vertices by an edge if and only if they belong to different subsets.

Let G be a simple graph with vertex set $\{v_1, \ldots, v_n\}$. The *adjacency matrix* of G, denoted by $A(G)$, is the $n \times n$ matrix such that the (i, j)-entry is 1 if v_i and v_j are adjacent, and otherwise is 0. Let $D(G)$ be the diagonal matrix $diag(d_1, \ldots, d_n)$, where d_i is the degree of v_i in G, for $i = 1, \ldots, n$. Many matrices are associated to graphs, see [1]-[8]. One of them is called the *signless Laplacian matrix* defined as $Q(G) = D(G) + A(G)$. The signless Laplacian matrix is real and symmetric, so all of its eigenvalues are real. By the signless Laplacian eigenvalues of G we mean the eigenvalues of the signless Laplacian matrix of G and show them as $q_1(G) \geq \cdots \geq q_n(G)$. Let $m_1, \ldots, m_t$ be some integers and $X = (m_1, m_2, \ldots, m_t)$. By $m_{[1]} \geq m_{[2]} \geq \cdots \geq m_{[t]}$ we mean the components of X in decreasing order. For example if $m_1 = 3$, $m_2 = 5$, $m_3 = 3$, $m_4 = 6$ and $X = (3, 5, 3, 6)$, then $m_{[1]} = 6$, $m_{[2]} = 5$, and $m_{[3]} = m_{[4]} = 3$.

2 Results

In this section we study the smallest signless Laplacian eigenvalue of graphs. The smallest signless Laplacian eigenvalue of G is denoted by $q'(G)$. It is known that $q'(G) = 0$ if and only if G has a bipartite component. We are interested in to study the following conjecture on q'.

[1]speaker

Conjecture 1..[3] Let $t \geq 3$ and let n be sufficiently large. If G is a K_{t+1}-free graph of order n and $G \neq T(n,t)$, then

$$q'(G) < q'(T(n,t)).$$

We disprove above conjecture by constructing some complete multipartite graphs.

Theorem 1.. Let $t \geq 3$ and $n_1, n_2, \ldots, n_t$ be some positive integers and $n = n_1 + \cdots + n_t$. Let $X = (n_1, n_2, \ldots, n_t)$ and $n_{[1]} \geq n_{[2]} \geq \cdots \geq n_{[t]}$ be the components of X in decreasing order. Then the following hold:

(i) If $n_{[1]} = n_{[2]}$, then $q'(K_{n_1,n_2,\ldots,n_t}) = n - 2n_{[1]}$.

(ii) If $n_{[1]} > n_{[2]}$, then $n - 2n_{[1]} < q'(K_{n_1,n_2,\ldots,n_t}) < n - n_{[1]} - n_{[2]}$.

Theorem 2.. Let t and n be some positive integers such that $n \geq 2t$ and let $n \equiv h$ (mode t). If $t \geq 4$ and $2 \leq h \leq t-2$, then there is at least one complete multipartite graph of order n and with t parts, say $K_{n_1,\ldots,n_t}$, except $T(n,t)$ such that

$$q'(K_{n_1,\ldots,n_t}) = q'(T(n,t)).$$

Corollary 1.. Let t and n be some positive integers such that $n \geq 2t$ and let $n \equiv h$ (mode t). Assume that $t \geq 4$ and $2 \leq h \leq t-2$. Since every t-complete multipartite graph is K_{t+1}-free, Theorem 2 implies that Conjecture 1 is not true.

Conjecture 2.. Let $t \geq 3$ and let n be sufficiently large. If $n \equiv 0$ (mode t) or $n \equiv 1$ (mode t) or $n \equiv t-1$ (mode t) and $G \neq T(n,t)$ is a K_{t+1}-free graph of order n, then

$$q'(G) < q'(T(n,t)).$$

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Inductive tools for maintaining connectivity in graphs and matroids

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Abstract

Tutte proved in 1961 that every 3-connected simple graph G, other than a wheel, has an edge whose deletion or contraction is both 3-connected and simple. Seymour (1980) and Negami (1982) independently strengthened Tutte's theorem by proving that, for any 3-connected simple proper minor H of G, we can delete or contract an edge from G to get a graph that, in addition to being both 3-connected and simple, maintains a minor isomorphic to H. Tutte generalized his theorem to matroids in 1966 while Seymour's original proof of his Splitter Theorem was done in the more general context of matroids. These theorems give us powerful inductive tools for working with graphs and matroids provided our structures are 3-connected. A number of authors, including Johnson and Thomas, and Geelen and Zhou, have sought corresponding results for graphs and matroids of higher connectivity. This talk will discuss the speaker's joint work with Carolyn Chun and Dillon Mayhew that finds analogues of the theorems of Tutte and Seymour for internally 4-connected binary matroids and hence for internally 4-connected graphs, where such graphs are 4-connected except for the possible presence of degree-3 vertices. The talk will assume no prior knowledge of matroid theory.

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On the sum of k largest Laplacian eigenvalues of graphs

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Abstract

Consider a simple graph $G(V, E)$ of order n, size m and having the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The adjacency matrix $A = (a_{ij})$ of G is a $(0, 1)$-square matrix of order n whose (i, j)-entry is equal to 1 if v_i is adjacent to v_j and equal to 0, otherwise. Let $D(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix associated to G, where $d_i = \deg(v_i)$, for all $i = 1, 2, \ldots, n$. The matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix and its eigenvalues are called the Laplacian eigenvalues of the graph G. Let $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$ be the Laplacian eigenvalues of G and $S_k(G) = \sum_{i=1}^{k} \mu_i$, $k = 1, 2, \ldots, n$ be the sum of k largest Laplacian eigenvalues of G. For any k, $k = 1, 2, \ldots, n$, A. Brouwer conjectured that $S_k(G) = \sum_{i=1}^{k} \mu_i \leq m + \binom{k+1}{2}$. We discuss the bounds for $S_k(G)$ and the recent developments of the Brouwer's conjecture. Further, we investigate analogous conjectures (of the Brouwer's type) in other types of graphs.

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Finite edge-transitive Cayley graphs, quotient graphs and Frattini groups

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Abstract

The edge-transitivity of a Cayley graph is most easily recognisable if the subgroup of affine maps preserving the graph structure is itself edge-transitive. These are the so-called normal edge-transitive Cayley graphs. Each of them determines a set of quotients which are themselves normal edge-transitive Cayley graphs and, and which are built from a very restricted family of groups (direct products of simple groups). We address the questions: how much information about the original Cayley graph can we retrieve from this special set of quotients? And can we ever reconstruct the original Cayley graph from them: if so, then how?

Our answers to these questions involve a type of relative Frattini subgroup determined by the Cayley graph, which has similar properties to the Frattini subgroup of a finite group Ill discuss this and give some examples. It raises many new questions about Cayley graphs.

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Several adjacency matrices for digraphs and their spectra

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Abstract

Several results on spectra of digraphs related to structural properties of directed graphs will be presented.

The question of which kind of adjacency matrix to use for digraphs seems even more important than for undirected graphs, since each such kind of matrix has its pros and cons. We will start with the "usual" adjacency matrix and relate it to some (older) results on digraphs with "walk-regular" properties. We next consider the recently becoming quite popular Hermitian adjacency matrix, which has the advantage of having real eigenvalues and present some results on spectral characterizations. Finally, we consider a variation of the Hermitian adjacency matrix (introduced recently by Mohar) that we call the Eisenstein matrix, and that can be used even for signed digraphs.

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Contributed Talks

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An update on some recent classifications of block designs admitting flag-transitive automorphism groups

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Abstract

In this talk, we give a survey on recent classification theorems of 2-designs admitting flag-transitive automorphism groups.

Keywords: 2-design, flag-transitive, automorphism group, primitive group, symmetric design.

Mathematics Subject Classification [2010]: Primary: 05B05; Secondary: 05B25, 20B25

1 Introduction

A 2-(v, k, λ) design $\mathcal{D}$ is a pair $(\mathcal{P}, \mathcal{B})$ with a set $\mathcal{P}$ of v points and a set $\mathcal{B}$ of b blocks such that each block is a k-subset of $\mathcal{P}$ and each two distinct points are contained in λ blocks. The *replication number* r of $\mathcal{D}$ is the number of blocks incident with a given point. We always assume that $\mathcal{D}$ is nontrivial, that is to say, $2 < k < v$. If $b = v$ (or equivalently $r = k$), the design $\mathcal{D}$ is called *symmetric*. An *automorphism* of $\mathcal{D}$ is a permutation on $\mathcal{P}$ which maps blocks to blocks and preserving the incidence. The *full automorphism* group $\mathrm{Aut}(\mathcal{D})$ of $\mathcal{D}$ is the group consisting of all automorphisms of $\mathcal{D}$. A *flag* of $\mathcal{D}$ is a point-block pair (α, B) such that $\alpha \in B$. A group $G \leqslant \mathrm{Aut}(\mathcal{D})$ is called *flag-transitive* if G acts transitively on the set of flags. The group G is said to be *point-primitive* if G acts primitively on $\mathcal{P}$. Further notation and definitions in both design theory and group theory are standard and can be found, for example in [7, 11].

2 Main Results

As a main part of this talk, we present some new results on automorphism groups of 2-designs acting transitively on the set of flags of the designs. We, in particular, introduce all such 2-designs with prime replication numbers [2]. We also give a list of all families of flag-transitive 2-designs whose replication number is coprime to λ with an exception in one dimensional affine space, see [1, 4, 5, 8, 9, 10, 12, 13] and therein references. Another part of this talk is devoted to presenting a framework in order to classifying flag-transitive automorphism groups of symmetric designs with λ prime [3, 6].

[1]speaker

[2]The main results presented in this talk are part of recent studies with my colleague Ashraf Daneshkhah, and my Ph.D. students Mohsen Bayat, Jalal Choulaki and Fatemeh Mouseli at Department of Mathematics, Faculty of Science, Bu-Ali Sina University, Hamedan, Iran.

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A note on dominating circuits of Splitting off matroids

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Abstract

In 1971, Nash-Williams proved that if G is a simple 2-connected graph on n vertices having minimum degree at least 1/3(n+2), then any longest cycle C in G is also edge-dominating; that is, each edge of G has at least one end-vertex incident with a vertex of C. We say that a circuit C in a matroid M is dominating if each component of M/C has rank at most one. In this paper, we generalize the dominating circuits in matroids and also we prove that all dominating circuits is not dominating circuits in splitting off matroids.

Keywords: Dominating circuit, Binary, Contraction, rank, Hamilton cycle
Mathematics Subject Classification [2010]: 05B35

1 Introduction

There are numerous theorems in graph theory relating a minimum degree condition on vertices of a graph to the length of a longest cycle. The first such result is a seminal theorem of Dirac[2]:

Theorem 1.1. *(Dirac). Let G be a simple graph of order $n \geq 3$ having minimum degree at least $n/2$.Then G has a Hamilton cycle.*

Since Diracs theorem, there has been a steady flow of similar theorems about cycles in graphs. The concept of a cycle in a graph corresponds to the more general concept of a circuit in a matroid. While some properties of cycles in graphs easily extend to circuits in matroids, there are other properties for which there is no simple answer. With Diracs theorem in mind, Welsh[10] posed the following lemma:

Lemma 1.2. *(Welsh) If M is a simple connected regular matroid and every cocircuit has at least $1/2(r(M)+1)$ elements, then M has a circuit of size r(M)+1.*

In two papers[6], Hochstttler and Jackson verified the above conjecture of Welsh by using a decomposition theorem of Seymour[11] for regular matroids. Various other authors have exploited Seymours theorem for the purpose of extending known theorems about graphs to regular matroids or proving new theorems(see[1,3,4,5,7,8]).With proofs of this type, one must consider the cases of graphic and cographic matroids separately. In contrast, the proof we give is purely matroid-theoretic. While no graph theory is used, many of the ideas introduced owe their intuitive basis to the properties of cycles and cocycles in graphs.

Graph theorists have also been interested in conditions which guarantee the existence of a so called dominating cycle, a generalization of a Hamilton cycle. A cycle in a graph is dominating if every edge of the graph is incident with at least one vertex of the cycle. The dominating cycle counterpart to Diracs theorem is the following result of Nash-Williams[9]:

[1]speaker

Theorem 1.3. *(Nash-Williams,1971). Let G be a simple 2-connected graph of order* $n \geq 3$*. If G has minimum degree at least* $1/3(n+2)$*, then any longest cycle in G is a dominating cycle.*

Since 1971, there have been many other variations and strengthenings of Nash-Williams result for graphs (seeVoss[14]), but none thus far for matroids. In this paper, I show, using only basic concepts and ideas involving matroids, that Nash-Williams theorem can be extended in a natural way to regular matroids.

Throughout this paper, C(M) (respectively $C^*(M)$) shall denote the set of circuits (respectively cocircuits) of a matroid M. Let M be a matroid and let $X \subseteq E(M)$. A subset $Y \subseteq E(M)\ X$ is called an X-bridge if Y is a component of M/X. We define the order of Y to be $o(Y) = r_{M/X}(Y)$. We shall let $K(M/X)$ denote the set of X-bridges of M. We say that a circuit C in M is dominating if $o(Y) \leq 1$ for each C-bridge Y.

2 Splitting off matroids

Note that, if $v_1 = v_2$, then what the splitting off operation does is removing x and y; whereas if $v_1 \neq v_2$, then the splitting off operation is to add an edge z in such a way that {x, y , z} forms a triangle, and then delete x and y. Considering these facts, Shikare and Azadi[12]] defined the splitting off operation for binary matroids.

Definition 2.1. Let M be a binary matroid and $x, y \in E(M)$. Also M_{xy} be the matroid obtained from M as follows: if x, y is a parallel pair, $M_{xy} = M\backslash\{x, y\}$, and if $\{x, y\}$ is an independent set, add a new element z in such a way that {x, y , z} is a 3-element circuit, and then delete x and y. The transition from M to M_{xy} is called a splitting off operation.

Using Theorem 1, give alternative approach (in terms of the circuits of the matroids) to the splitting off operation in binary matroids.

Theorem 2.2. [13] *Let* $M = (S, \mathcal{C})$ *be a binary matroid on the set S together with the set* $\mathcal{C}$ *of circuits. Let* $x, y \in S$ *and z be an element that is not in S. We define*
$C_0 = \{C \in \mathcal{C} :$ *C contains neither of x and y* $\}$;
$C_1 = \{(C - \{x, y\}) \cup \{z\} : C \in \mathcal{C}$ *and* $x, y \in C\}$; *and*
$C_2 = \{((C_1 \cup C_2) - \{x, y\}) \cup \{z\} : C_1, C_2 \in \mathcal{C}, C_1 \cap C_2 = \emptyset, x \in C_1, y \in C_2$ *and* $C_1 \cup C_2$ *contains no circuit of M containing both x and y , or neither*$\}$.
Further, we define $C_{xy} = C_0 \cup C_1 \cup C_2$ *and* $S' = (S - \{x, y\}) \cup \{z\}$*. Then the pair* (S', C_{xy}) *is a binary matroid. We denote this matroid by the notation* M_{xy} *and call it the splitting off matroid.*

3 Main Results

Theorem 3.1. *Let M be a sample connect matroid. C be a dominating circuit of M and Y component of matroid* M/C *Then* $\forall Y \subseteq M/C; r(Y) \leqslant 1$ *if and only if all elements of E(M) other than elements of C formed circuits with element of C such that every circuit contains at most two elements of E(M) other than elements of C*

Proof. Let for all Y, components of matroid M/C; $r(Y) \leqslant 1$ and let there is a circuit such that contains more than two elements of E(M) other than elements of C. Then by contracting elements of C, matroid M/C contains a circuit contains more than two elements. This is a contradiction.

Now let all elements of E(M) other than elements of C formed circuits with element of C such that every circuit contains at most two elements of E(M) other than elements of C. If some circuits D_1 and D_2 meet in elements of E(M) other than elements of C. And let e, f and e, g be elements D_1 and D_2 not in C, respectively, then by contracting all elements of C, $\{e, f\}$ and $\{e, g\}$ are parallel set. Hence we have a rank -1 component. Also, other component in the matroid M/C have rank at most one. There fore all components Y of M/C ; $r(Y) \leqslant 1$ □

Theorem 3.2. *Let M be a sample and connect matroid. Then D is a dominating circuit of M if and only if number of rank-1 component* M/D *i equal to* $r(M) - |D| - 1$

Proof. Let D is a dominating circuit. Since for all subsets T of M, rank M/T is $r_{M/}(E-T) = r(M) - r(B_T)$. Therefor rank M/D is $r(M) - |D| - 1$. Now if number of rank-1 component M/D isn't $r(M) - |D| - 1$, then there is a component with rank more than one. It is a contradiction.

Now let number rank-1 component M/D is equal to $r(M) - |D| - 1$. Since rank M/D is equal to $r(M) - |D| - 1$ too. Thus we have no component with a rank more than one. □

Theorem 3.3. *Let M be a sample, connect and binary matroid and $x, y \in E(M)$, $z \notin E(M)$. Let C and D be minimal dominating circuits M such that C does not contain x and y but D contains both of x and y. If set of $\{x, y\}$ contains a cocircuit of M or the number of rank-1 components of $M/(D\ \{x, y\} \cup \{z\})$ be more than the number of rank-1 components of M/D. Then C isn't a dominate circuit of M_{xy}.*

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On Extension of Graphic Matroids

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Abstract

In this paper, we study a special case of Γ-extension of a binary matroid, that is Γ- extension of graphic matroid. We obtain some results concerning the weak order, bipartite, eularian of matroids and Γ-extension operation.

Keywords: binary matroid, Γ-extension of graphic matroid, bipartite matroid, Eulerian matroid.
Mathematics Subject Classification [2010]: 05B35

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On the total transmission irregularity of graphs

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Abstract

For the purpose of measuring the transmission irregularity of graphs, a new structural invariant named the total transmission irregularity is proposed for a graph G as

$$Irr_{Ttr}(G) = \frac{1}{2} \sum_{u,v \in V(G)} |\sigma_G(u) - \sigma_G(v)|,$$

where $\sigma_G(u)$ denotes the transmission of the vertex u in G which is the sum of distances between u and all other vertices of G and the summation runs over all ordered pairs of vertices of G. The aim of this paper is to investigate some basic mathematical properties of the total transmission irregularity and specially to study it for some families of product graphs.

Keywords: Vertex transmission, Transmission regular graph, Product graphs, Bound.
Mathematics Subject Classification [2010]: Primary: 05C12; Secondary: 05C76.

1 Introduction

In this paper, we consider simple connected finite graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *degree* $d_G(u)$ of the vertex $u \in V(G)$ is the number of edges incident to u. The *distance* $d_G(u, v)$ between the vertices $u, v \in V(G)$ is defined as the length of any shortest path in G connecting u and v. The *transmission* (or *status*) $\sigma_G(u)$ of a vertex $u \in V(G)$ is defined as the sum of distances between u and all other vertices v of G, *i.e.*, $\sigma_G(u) = \sum_{v \in V(G)} d_G(u, v)$.

A *graph invariant* is a numerical value associated to a graph which is structurally invariant.

A graph G is said to be *regular* if all its vertices have equal degrees, otherwise it is said to be *irregular*. In order to measure the irregularity of graphs, *i.e.*, how far is a graph from being regular, Abdo *et al.* [2] introduced the *total irregularity* of a graph G as

$$irr_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} \left| d_G(u) - d_G(v) \right|,$$

where the summation goes over all ordered pairs of vertices of G. It is easy to see that $irr_t(G) = 0$ if and only if G is regular. We refer the reader to [1, 4, 5] for some recent results on irregularity measures.

A graph G is said to be *transmission regular* if all its vertices have the same transmission, otherwise it is said to be *transmission irregular*. In order to quantify the transmission irregularity in graphs, *i.e.*, the deviation of a graph from being transmission regular, we propose here a structural invariant called the *total transmission irregularity* which is defined for a graph G as

$$Irr_{Ttr}(G) = \frac{1}{2} \sum_{u,v \in V(G)} |\sigma_G(u) - \sigma_G(v)|.$$

[1] speaker

It is easy to see that $Irr_{Ttr}(G) = 0$ if and only if G is transmission regular. Note that, this invariant has a parallel form to the total irregularity of graphs by replacing the vertex degrees with the vertex transmissions.

The aim of this paper is to investigate some basic mathematical properties of the total transmission irregularity and specially to study it for some families of product graphs.

2 Main Results

In this section, we derive exact expressions or upper bounds for the total transmission irregularity of several families of product graphs. We denote the components of each graph product by G_1 and G_2 which are assumed to be nontrivial finite simple connected graphs of order n_1 and n_2 and size m_1 and m_2, respectively. We refer the reader to [9] for detailed exposition on graph products and to [1, 3, 6, 7, 8, 10] for more information on computing graph invariants of product graphs.

2.1 Join

The *join* $G_1\nabla G_2$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph union $G_1 \cup G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$.

Lemma 2.1. *The distance between the vertices $u, v \in V(G_1\nabla G_2)$ is given by*

$$d_{G_1\nabla G_2}(u,v) = \begin{cases} 0 & \text{if } u = v, \\ 1 & \text{if } uv \in E(G_1) \text{ or } uv \in E(G_2) \text{ or } [u \in V(G_1),\ v \in V(G_2)], \\ 2 & \text{otherwise.} \end{cases}$$

We first consider the case when one of the components in join is single vertex. For a given graph G, the graph $K_1\nabla G$ is called the *suspension* of G, where K_1 denotes a single vertex. In the following theorem, we present an exact expression for the total transmission irregularity of the suspension of a given graph.

Theorem 2.2. *Let G be a graph of order n and size m. Then*

$$Irr_{Ttr}(K_1\nabla G) = irr_t(G) + n(n-1) - 2m.$$

Now, we tackle the case when the components in join are not singleton.

Theorem 2.3. *Under the condition $n_1 \geq n_2$, the total transmission irregularity of $G_1\nabla G_2$ satisfies the following inequality:*

$$Irr_{Ttr}(G_1\nabla G_2) \leq irr_t(G_1) + irr_t(G_2) + n_2(n_1-1)(n_1-2).$$

Moreover, the bound is sharp.

2.2 Corona product

Let G_1 be a nontrivial graph of order n_1. The *corona product* $G_1 \circ G_2$ of graphs G_1 and G_2 is the graph obtained by taking one copy of G_1 and n_1 copies of G_2, and by joining each vertex of the ith copy of G_2 to the ith vertex of G_1, for $i = 1, 2, \ldots, n_1$. We denote the copy of G_2 related to the vertex $x \in V(G_1)$ by $G_{2,x}$.

Lemma 2.4. *The distance between the vertices $u, v \in V(G_1 \circ G_2)$ is given by*

$$d_{G_1\circ G_2}(u,v) = \begin{cases} d_{G_1}(u,v) & \text{if } u,v \in V(G_1), \\ 1 & \text{if } uv \in E(G_{2,x}), \\ 2 & \text{if } u,v \in V(G_{2,x}),\ uv \notin E(G_{2,x}), \\ d_{G_1}(u,x)+1 & \text{if } u \in V(G_1),\ v \in V(G_{2,x}), \\ d_{G_1}(x,y)+2 & \text{if } u \in V(G_{2,x}),\ v \in V(G_{2,y}),\ x \neq y. \end{cases}$$

Theorem 2.5. *The total transmission irregularity of $G_1 \circ G_2$ satisfies the following inequality:*

$$Irr_{Ttr}(G_1 \circ G_2) \leq (n_2+1)^3 Irr_{Ttr}(G_1) + {n_1}^2 irr_t(G_2) + {n_1}^2(n_1{n_2}^2 + n_1n_2 - 2n_2 - 2m_2).$$

The equality holds if and only if G_1 is transmission regular.

2.3 Disjunction

The *disjunction* $G_1 \vee G_2$ of graphs G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$ in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent whenever $u_1v_1 \in E(G_1)$ or $u_2v_2 \in E(G_2)$.

Lemma 2.6. *The distance between the vertices* $(u_1, u_2), (v_1, v_2) \in V(G_1 \vee G_2)$ *is given by*

$$d_{G_1 \vee G_2}((u_1, u_2), (v_1, v_2)) = \begin{cases} 0 & \text{if } u_1 = v_1,\ u_2 = v_2, \\ 1 & \text{if } u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2), \\ 2 & \text{otherwise.} \end{cases}$$

Theorem 2.7. *The total transmission irregularity of* $G_1 \vee G_2$ *satisfies the following inequality:*

$$Irr_{Ttr}(G_1 \vee G_2) \leq n_2\big({n_2}^2 + 2m_2\big)irr_t(G_1) + n_1\big({n_1}^2 + 2m_1\big)irr_t(G_2).$$

2.4 Symmetric difference

The *symmetric difference* $G_1 \oplus G_2$ of graphs G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$ in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent whenever $u_1v_1 \in E(G_1)$ or $u_2v_2 \in E(G_2)$, but not both.

Lemma 2.8. *The distance between the vertices* $(u_1, u_2), (v_1, v_2) \in V(G_1 \oplus G_2)$ *is given by*

$$d_{G_1 \oplus G_2}((u_1, u_2), (v_1, v_2)) = \begin{cases} 0 & \text{if } u_1 = v_1,\ u_2 = v_2, \\ 1 & \text{if } u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2), \text{ but not both,} \\ 2 & \text{otherwise.} \end{cases}$$

Theorem 2.9. *The total transmission irregularity of* $G_1 \oplus G_2$ *satisfies the following inequality:*

$$Irr_{Ttr}(G_1 \oplus G_2) \leq n_2\big({n_2}^2 + 4m_2\big)irr_t(G_1) + n_1\big({n_1}^2 + 4m_1\big)irr_t(G_2).$$

2.5 Cartesian product

The *Cartesian product* $G_1 \square G_2$ of graphs G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$ in which two vertices (v_1, v_2) and (u_1, u_2) are adjacent whenever $[u_1 = v_1$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2$ and $u_1v_1 \in E(G_1)]$.

Lemma 2.10. *The distance between the vertices* $(u_1, u_2), (v_1, v_2) \in V(G_1 \square G_2)$ *is given by*

$$d_{G_1 \square G_2}((u_1, u_2), (v_1, v_2)) = d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2).$$

Theorem 2.11. *The total transmission irregularity of* $G_1 \square G_2$ *satisfies the following inequality:*

$$Irr_{Ttr}(G_1 \square G_2) \leq {n_2}^3 Irr_{Ttr}(G_1) + {n_1}^3 Irr_{Ttr}(G_2).$$

The equality holds if and only if G_1 *or* G_2 *is transmission regular.*

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Some results on annihilator graph

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Abstract

In this talk, a number of results on annihilator graph of a commutative ring R will be discussed.

Keywords: zerodivisor graph, annihilator graph
Mathematics Subject Classification [2010]: Primary 13A15; Secondary 13B99; 05C99

1 Introduction

Beck-Anderson-Livingston *zero-divisor graph* of R is the (simple) graph $\Gamma(R)$ with vertices $Z(R) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. In 2014, Badawi introduced the *annihilator graph of* R. The annihilator graph of R is the (undirected) graph $AG(R)$ with vertices $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$ (Recall if m in R, then $ann_R(m) = \{y \in R \mid ym = 0\}$). It follows that each edge (path) of the classical zero-divisor of R is an edge (path) of $AG(R)$. In this talk, we state a number of results about some graphs as in [1], [2], [3] and [4]. Among many results, we state the following.

2 Main Results

Theorem 2.1. *Let R be a commutative ring with $|Z(R)^*| \geq 2$. Then $AG(R)$ is connected and $diam(AG(R)) \leq 2$.*

Theorem 2.2. *Let R be a reduced commutative ring and suppose that $AG(R) \neq \Gamma(R)$. Then $gr(AG(R)) = 3$. Furthermore, there is a cycle C of length three in $AG(R)$ such that each edge of C is not an edge of $\Gamma(R)$.*

The following is an example of a non-reduced commutative ring R where $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$, but every path in $AG(R)$ of length two from x to y is also a path in $\Gamma(R)$.

Example 2.3. Let $R = \mathbb{Z}_8$. Then $2 - 6$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$. Now $2 - 4 - 6$ is the only path in $AG(R)$ of length two from 2 to 6 and it is also a path in $\Gamma(R)$. Note that $AG(R) = K^3$, $\Gamma(R) = K^{1,2}$, $gr(\Gamma(R)) = \infty$, $gr(AG(R)) = 3$, $diam(\Gamma(R)) = 2$, and $diam(AG(R)) = 1$.

Theorem 2.4. *Let R be a commutative ring and suppose that $AG(R) \neq \Gamma(R)$. Then the following statements are equivalent:*

1. *$gr(AG(R)) = 4$;*

[1] speaker

2. $gr(AG(R)) \neq 3$;

3. *If* $x - y$ *is an edge of* $AG(R)$ *that is not an edge of* $\Gamma(R)$ *for some distinct* $x, y \in Z(R)^*$, *then there is no path in* $AG(R)$ *of length two from* x *to* y;

4. *There are some distinct* $x, y \in Z(R)^*$ *such that* $x - y$ *is an edge of* $AG(R)$ *that is not an edge of* $\Gamma(R)$ *and there is no path in* $AG(R)$ *of length two from* x *to* y;

5. R *is ring-isomorphic to either* $\mathbb{Z}_2 \times \mathbb{Z}_4$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$.

Theorem 2.5. *Let* R *be a commutative ring such that* $AG(R) \neq \Gamma(R)$. *Then* $gr(AG(R)) \in \{3, 4\}$.

Theorem 2.6. *Let* R *be a ring. Then* $AG(R)$ *is a complete bipartite graph if and only if one of the following statements holds:*

1. $Nil(R) = \{0\}$ *and* $|Min(R)| = 2$;

2. $Nil(R) \neq \{0\}$ *and either* $AG(R) = K^{1,n}$ *or* $AG(R) = K^{2,3}$, *where* $1 \leq n \leq \infty$.

Theorem 2.7. *Let* R *be a ring. Then* $AG(R)$ *is a complete bipartite graph if and only if one of the following statements holds:*

1. $Nil(R) = \{0\}$ *and* $|Min(R)| = 2$;

2. $Nil(R) \neq \{0\}$ *and either* $AG(R) = K^{1,n}$ *or* $AG(R) = K^{2,3}$, *where* $1 \leq n \leq \infty$.

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Burning numbers of some graphs from designs

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Abstract

The burning number of a graph is a quantity which measures the speed of the spread of contagion in a graph. In this paper, we study the burning number of a family of graphs associated with a family of block designs.

Keywords: Burning number, Block Design, Block Intersection graph.

1 Introduction

Let t, k, v and λ be nonnegative integers such that $0 < t \leq k \leq v$ and $\lambda \geq 0$. Moreover, let V be a v- set. The set all of the i-subsets of V is denoted by $P_i(V)$. A piar $D = (V, \beta)$ is called a $t-(v,k,\lambda)$ design, if B is a family of k-subset of V(called blocks), such that every t-subset of V appears in exactly λ blocks [1]. The number of blocks in D and the number of repetitions of points are denoted by b and r, respectively. A graph is a pair $G = (V, E)$ where $E \in P_2(V)$. we call V and E the vertex set and the edge set of G, respectively. Hence two vertices u and v are adjacent or neighbor in G if $u, v \in E$. Let $v \in V$. The degree of v denoted by deg(v) is the number of vertices u such that $\{u, v\} \in E$[3]. A dominating set S for a graph G is a subset of vertices $S \subset V(G)$ such that every vertex not in S is adjacent to at least one element of S. The domination number of G, denoted by $\gamma(G)$, is the minimum size of a dominating set in G[5]. To define the notion of burning number of a graph first we define the process of burning nodes(vertices) of a graph in individual rounds. That is each node either is burned or is unburned. If a node is burned, then it remains burned until the last round. In each round, we choose one additional unburned node and burn (if such a node is available). Once a node is burned in round t, then in round t + 1, each of its unburned neighbors burn. The process ends when all nodes are burned. The burning number of a graph G, denoted by b(G), is the minimum number of rounds of burning process needed to burn all nodes. Suppose that all nodes of a graph G burn in k rounds ($k \leq b(G)$) then we donote by x_i the set of all nodes which is burned in i-th round where $1 \leq i \leq k$ and we call the sequence $(x_1, x_2, ..., x_k)$ as a burning sequence for G. Let G be a graph and v be a node of G, then the eccentricity of v is defined as $max\{d(v, u) : u \in V(G)\}$. The radius and diameter of G is the minimum and maximum eccentricity over the set of all nodes in G, respectively[4].

Theorem 1.1. *[4] For a path P_n of order n, we have that*

$$b(G) = \lceil \sqrt{n} \rceil.$$

Theorem 1.2. *[4] For any graph G with radius r and diameter d, we have*

$$b(G) = \left\lceil \sqrt{d(G)+1} \right\rceil \leq b(G) \leq r(G) + 1.$$

[1]speaker

2 Main Results

Let D be a design with point set X and block set B. The incidence graph of D is dened by $G_D = (X \cup B, E)$ where $(x, B) \in E \iff x \in B$. The graph G_D is a bipartite graph such that two sets of points and blocks of D represent the two classes of independent vertices and the adjacency relation denote the incidence structure of D[5].

Theorem 2.1. *Let D be a 2-(v,k,1) design with $v \neq k$ and $v \geq 8$, then $b(G_D) = 4$.*

Proof. Let $X = \{x_1, ..., x_v\}$, $B = \{B_1, ..., B_b\}$, then it is clear that a sequence of the following form is a burning sequence for G_D:

$$(x_1, B_1, x_2, B_2) \quad where \quad x_1x_2 \notin B_i, i = 1, 2.$$

If we show that no sequence of length 3 is not a burning sequence then the result is obtained. There are eight sequences of length 3 non are a burning set and this can be shown with a similar argument. Let for example (B_1, B_2, x_1), where B_1 and B_2 have no intersection. If (B_1, B_2, x_1) is a burning sequence then we have $v = 2k$ or $v = 2k + 1$ which implies to $k = 2$ or $k = 3$ since r is an integer. This implies to $v = 4$ or $v = 5$ or $v = 7$ which is a contradiction since $v \geq 8$. □

Given a combinatorial design D with block set B, its traditional block intersection graph G_{BD} is the graph having vertex set B such that two vertices (blocks) b_1 and b_2 are adjacent if and only if b_1 and b_2 have no empty intersection[2].

Proposition 2.2. *If D be a symmetric $t - (v, k, 1)$ design, then $b(G_{BD}) = 2$.*

Proof. since D is symmetric then any two blocks of D have intersection of size one hence G_{BD} is a complete graph, and we have

$$b(G_{BD}) = 2.$$

□

Let there exist a $s - (v, k, 1)$ design with $0 < s < k < v$ and $t \geq 2$ and let

$$X_1 = 1, 2, ..., v, X_2 = v, v + 1, ..., 2v - 1, ..., X_t = v(t - 1) - (t - 2), ..., vt - (t - 1).$$

Let D_i be a $s - (v, k, 1)$ design on X_i. Now consider the set of blocks $D_i's$ as vertices a new graph such that two vertices (blocks) are adjacent if both have a common element. Clearly this graph is connected since X_i and X_{i+1} have a common element. We call this graph a union block intersection graph and is denoted by G_{UD} . If the existence $s - (v, k, 1)$ design is symmetric, we may consider all $D_i's$ be symmetric then we will denote the above graph with G_{UDS}. The order of G_{UD} is bt.

Theorem 2.3. *diam $G_{UD} = t + 1$.*

Proof. The maximum distance between vertices of G_{UD} belongs to the blocks of the D_1 and D_t designs, neither of them have points v and $v(t - 1) - (t - 2)$. Suppose that $b_0 = x_1x_2...x_k$ is a block of D_1, where $v \notin b_0$ and also $b_{t+1} = y_1y_2...y_k$ is a block of the D_t, where $v(t-1)-(t-2) \notin b_{t+1}$. Accordingly, we consider a block such as b_1 in the design D_1 containing v, which contain at least one element of b_0. Clearly b_1 and b_0 are adjacent. The block b_2 in the D_2 containing {v,2v-1} is adjacent with b_1. If we continue this process we may consider a block b_t in D_t containing two elements $\{y_1, v(t - 1) - (t - 2)\}$. This block is also adjacent to the block b_{t+1}. Therefore, a path of order t + 1 as $b_0, b_1, ..., b_{t+1}$ will be obtained between two vertices b_0 and b_{t+1}, which is at the same time the shortest path between these two vertices. □

Lemma 2.4. *Let S be a dominating set of G and let G_S be the induced subgraph of G. Then*

$$b(G) \leq b(G_S) + 1.$$

Proof. Since all vertices of G\ G_S are connected to the vertices of G_S, so in any burning process of G they burn at most the next round after burning G_S. This means that clearly

$$b(G) \leq b(G_S) + 1.$$

□

Theorem 2.5. *Considering the graphs G_{UD} and G_{UDS} we have:*

a. $b(G_{UDS}) \leq 1 + \lceil\sqrt{t}\rceil$.
b. $b(G_{UD}) \leq 2 + \lceil\sqrt{t}\rceil$.

Proof. **a.**Suppose that b_1 is a block of design D_1 containing v. We appoint a block b_2 of D_2 containing elements $\{v, 2v-1\}$. Similarly, we appoint a block b_3 of D_3, containing elements $\{2v-1, 3v-2\}$. We may continue this process to reach the block b_t of D_t, containing $v(t-1)-(t-2)$. The path $P_t = b_1, b_2, ..., b_t$ is a path of order t, then by theorem 2 we have $b(P_t) = \lceil\sqrt{t}\rceil$. Now by Lemma 1 we conclude that:

$$b(G_{UDS}) \leq 1 + \lceil\sqrt{t}\rceil.$$

b.We do the same argument as in Part a and select the block b_i of D_i. All other blocks in D_i are connected to b_i with a path with a maximum length of 2. If we add all the adjacent blocks of b_i to this path, we have a dominating connected set with size rt for G_{UD}. The path P_t defined in part a is part of this dominating set. Now if we start burning P_t then G_{UD} would burn with appointing another two unburned vertices. Therefore,

$$b(G_{UD}) \leq 2 + \lceil\sqrt{t}\rceil.$$

□

Corollary 2.6. *Considering G_{UDS} we have the following bounds:*
a. $\lceil\sqrt{t+2}\rceil \leq b(G_{UDS}) \leq \lceil\sqrt{t}\rceil + 1$.
b. $\lceil\sqrt{t+2}\rceil \leq b(G_{UD}) \leq \lceil\sqrt{t}\rceil + 2$.

Corollary 2.7. *Considering G_{UDS}, If $t = m^2$ and $t = m^2 - 1$, then we have*

$$b(G_{UDS}) = \lceil\sqrt{t}\rceil + 1.$$

Note. If in above corollary $m^2 + 1 \leq t \leq (m+1)^2$ then we guess

$$b(G_{UDS}) = m + 2.$$

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Dot product dimension of unicyclic graphs

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Abstract

A graph $G = (V(G), E(G))$ is called a k-dot product graph if there is a function $f : V(G) \longrightarrow \mathbb{R}^k$ such that for any two distinct vertices u and v, one has $f(u).f(v) \geq 1$ if and only if $uv \in E(G)$. The minimum value k such that G is a k-dot product graph, is called the dot product dimension $\rho(G)$ of G. These concepts for the first time were introduced by Fiduccia, Scheinerman, Trenk and Zito. In this paper, we determine the dot product dimension of unicyclic graphs.

Keywords: k-dot product representation, k-dot product dimension.
Mathematics Subject Classification [2010]: Primary: 05C62

1 Introduction

In this paper, we study a geometric representation of graphs. If we assign to a vertex v of G a set S_v such that for any two distinct vertices u and v, $uv \in E(G)$ if and only if $S_v \cap S_u \neq \varnothing$. then we get an intersection representation. The intersection number of a graph is the minimum size of the union of the associated sets S_v to the vertices [2]. Fiduccia et al. extended the concept of intersection representation to dot product representation [2]. Let G be an arbitrary simple, and undirected graph of order n and k be a positive integer. Then G is called a k-dot product graph, if there exists a function $f : V(G) \longrightarrow \mathbb{R}^k$ such that for any two distinct vertices v and u, $f(u).f(v) \geq 1$ if and only if $uv \in E(G)$. We write $\boldsymbol{v}$ for $f(v)$. The dot product dimension of G is the minimum value k such that G is a k-dot product graph and is denoted by $\rho(G)$. We assume $\rho(tK_1) = 0$ where tK_1 is the graph which consists of t isolated vertices. Every graph has a k-dot product representation for an integer k and it has finite dot product dimension; see [2].

A 1-dot product graph has at most two nontrivial connected components that each of them is a threshold graph [2]. Dot product representations of graphs were studied from different aspects, especially as models for social networks [8, 11, 10, 7, 6, 9]. Bailey showed how dot product networks can be used in several applications in ecology [1]. Kang and Muller proved that recognizing graphs of any fixed dot product dimension $d \geq 2$ is NP-hard [4]. Any graph G is a $|E(G)|$-dot product graph and $\rho(G) \leq n - 1$ [2]. Fiduccia et al. investigated the dot product dimension of complete multipartite, bipartite and interval graphs [2]. They proved that every tree is a 3-dot product graph and there exist trees which are not 2-dot product graphs [2]. An outerplanar graph is a graph that has a planar drawing for which all vertices belong to the outer face of the drawing. Every planar graph and outerplanar graph are 4 and 3 dot product graphs, respectively. There exist a planar graphs which are not 3-dot product and some outerplanar graphs which are not 2-dot product [3]. Every cycle is a 2-dot product graph [2].

Feduccia et al. [2] conjectured that every graph G on n vertices satisfied $\rho(G) \leq \lfloor n/2 \rfloor$, and they proved this conjecture for every bipartite graph.

A unicyclic graph is a connected graph that contains exactly one cycle. We denote by $g(G)$ the girth of the graph G. In this paper, we obtain the exact dimensions of unicyclic graphs.

[1]speaker

2 The unicyclic graphs with girth greater than 5

Theorem 2.1. [5, Theorem 3] *Every outerplanar graph is a 3-dot product graph, and there are outerplanar graphs that are not 2-dot product graphs.*

Let G be a unicyclic graph with $g(G) \geq 6$. In this section, we determine $\rho(G)$.

Theorem 2.2. *Let G be a unicyclic graph with $g(G) \geq 6$. Then*

(i) $\rho(G) = 2$, *if* $G = C_n$ *or* $G = \mathcal{F}$ *is the graph depicted in Figure 1, and*

(ii) $\rho(G) = 3$, *otherwise.*

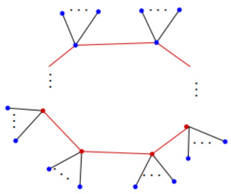

Figure 1: The graph $\mathcal{F}$ of Theorem 2.2.

3 The unicyclic graphs with girth 5

At first, we determine the dimension of the unicyclic graphs with girth 5 for some special cases in Lemma 3.1, and then determined the dot product dimension of all unicyclic graphs with girth 5 in Theorem 3.2.

Lemma 3.1. *Let $m, n \in \mathbb{N}$. The dot product dimension of the following unicyclic graphs is* 2

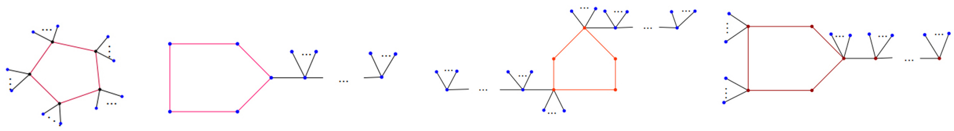

Figure 2: The graphs of Lemma 3.1 .

Now by Lemma 3.1, we can obtain the dot product dimension of the unicyclic graphs of girth 5.

Theorem 3.2. *Let G be a unicyclic graph with $g(G) = 5$. Then*

(i) $\rho(G) = 2$, *if G is C_5 or one of The graphs of Lemma 3.1, and*

(ii) $\rho(G) = 3$, *otherwise.*

4 The unicyclic graphs with girth 4

In this section, we determine the dimension of the unicyclic graphs with girth 4 for some special cases in Lemma 4.2 and Theorem 4.1 and finally determine the dot product dimension of all of unicyclic graphs with girth 4 in Theorem 4.3.

Theorem 4.1. *Let $G = \mathcal{H}_1$ be a graph that is obtained from C_4 by joining two adjacent vertices of C_4 to two leaves. Then* $\rho(G) = 3$.

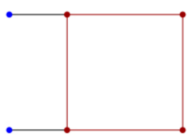

Figure 3: The graph $\mathcal{H}_1$ of Theorem 4.1.

Lemma 4.2. *The dot product dimension of the following unicyclic graphs is* 2.

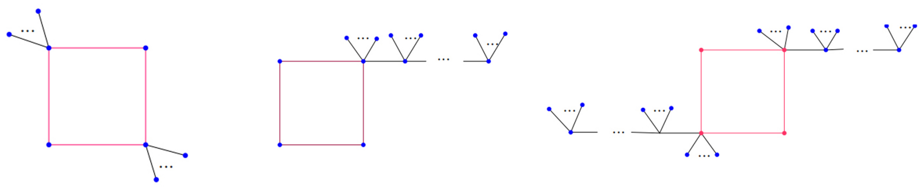

Figure 4: The graphs of Lemma 4.2.

By Lemma 4.2 and Theorem 4.1, we state the following Theorem on unicyclic graphs with girth 4.

Theorem 4.3. *Let G be an unicyclic graph with $g(G) = 4$. Then*

(i) $\rho(G) = 2$ *, if G is C_4 or one of The graphs of Lemma 4.2, and*

(ii) $\rho(G) = 3$*, otherwise.*

5 The unicyclic graphs with the girth 3

Let k be a positive integer and $S^{k-1} = \{u \in R^k : |u| = 1\}$. A cap of S^{k-1} is a nonempty intersection of S^{k-1} with a closed half-space.

Definition 5.1. [2] We say G is a k-cap-capture graph if we can assign to each vertex $v \in V(G)$ a cap C_v on S^{k-1} so that
(1) when $v \sim w$ either the center of C_v is contained in C_w, or the center of C_w is contained in C_v,
(2) when $v \nsim w$ we have $C_v \cap C_w = \varnothing$.

For $u \in \mathbb{R}^2$, let us denote by arg u the angle between the initial side (is always the positive x-axis) and the vector u.

Theorem 5.2. [2, Theorem 22] *If G is a k-cap-capture graph, then G is a k-dot product graph.*

Lemma 5.3. *Let $G = \mathcal{K}_1$ be a graph obtained from C_3 by joining every vertex of C_3 to one P_2. Then* $\rho(G) = 3$.

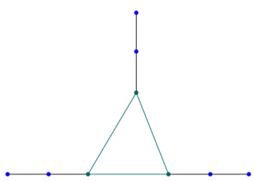

Figure 5: The graph $\mathcal{K}_1$ of Lemma 5.3.

Lemma 5.4. *The following unicyclic graphs have 2-dot product representations.*

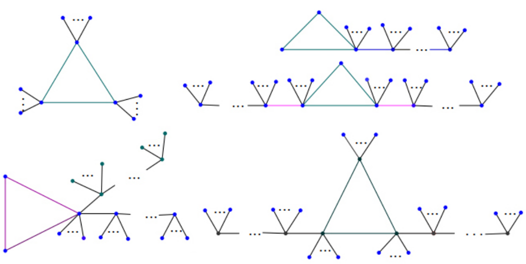

Figure 6: The graphs of Lemma 5.4.

By Lemmas 5.3 and 5.4, we state the following theorem on unicyclic graphs with girth 3.

Theorem 5.5. *Let G be an unicyclic graph with girth 3. Then*

(i) $\rho(G) = 1$ *, if* $G = \mathcal{C}$*,*

(ii) $\rho(G) = 2$ *, if G is one of The graphs of Lemma 5.4, and*

(iii) $\rho(G) = 3$*, otherwise.*

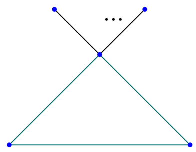

Figure 7: The graph $\mathcal{C}$ of Theorem 5.5.

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The Weight Hierarchy of some BCH Codes

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Abstract

Let C be a linear code over a finite field, the r-th generalized Hamming weight (r-th GHW) is defined the minimum of the support size of r-dimensional sub-codes of C and is denoted by $d_r(C)$. Calculation of GHW is an applicable and interesting topic in coding theory. In this paper, we will study the generalized Hamming weight(GHW) for some BCH codes.

Keywords: BCH Code, Generalized Hamming weight, Weight hierarchy
Mathematics Subject Classification [2010]: 05C69, 94B65

1 Introduction

For the first time, V. Wei introduced the concept of Generalized Hamming Weight in his famouse paper [7]. After Wei, several authors studied on this topic. For more information see [1], [2], [3] and [4] .
Let F_q be a field of order q and $n \in N$. An $[n,k,d]_q$ linear code, C, means that C is a subspace of F_q^n of dimension k, and minimum distance d. The Hamming weight of C, denoted by $wt(C)$, is the smallest of the weights of the nonzero codewords of C. Clearly, we have $d(C) = wt(C)$ for a linear code C.
For a subspace C of F_q^n, the support of C, denoted by $supp(C)$, is defined as follows

$$supp(C) = \{i : \exists (v_1, v_2, \ldots, v_n) \in C; v_i \neq 0\}.$$

Also, we define the r-th generalized Hamming weight (GHW) as follows

$$d_r = d_r(C) = min\{\|D\| : D \subset C, dim(D) = r\}$$

where $\|D\| = |supp(D)|$.
The set containing all of GHW s of a code is called the Weight Hierarchy (WH) for that code.

Let α be a primitive element of F_{q^m} and denote by $M^{(i)}(x)$ the minimal polynomial of α^i with respect to F_q. A (primitive) BCH code over F_q of length $n = q^m - 1$ with designed distance δ is a q-ary cyclic code generated by $g(x) := lcm(M^{(a)}(x), M^{(a+1)}(x), ..., M^{(a+2)}(x))$ for some integer a. Furthermore, the code is called narrow-sense if $a = 1$ (see [5]).

In this paper we shall study the generalized Hamming weight for some BCH codes.

2 Main Results

In this section, we study some necessary theorems.

Theorem 2.1. *(Monotonicity)*[1] *For an $[n,k]$ linear code C with $k > 0$, we have*

$$1 \leq d_1(C) < d_2(C) < ... < d_k(C) \leq n.$$

[1]speaker

Theorem 2.2. [7] *Let H be a parity check matrix of C, and let H_i, $1 \leq i \leq n$, be its column vectors. Let $< H_i : i \in I >$ denote the space generated by those vectors. Then*

$$d_r = min\{|X| : |X| - rank(< H_i > i \in X) = r\}.$$

Theorem 2.3. [1] *If C is an $[n, k, d]$ linear code over F_q, then*

$$d_r \leq n - k + r.$$

Theorem 2.4. *Let α be a primitive element of F_2^m. Then the weight hierarchy of a narrow sense binary BCH code (say B_m) is as follows*

$WH(B_2) = 3, WH(B_3) = 3, 5, 6, 7$ *and*

$$WH(B_4) = \begin{cases} \{3, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15\} & \delta = 2, 3, \\ \{5, 8, 10, 11, 13, 14, 15\} & \delta = 4, 5, \\ \{7, 10, 13, 14, 15\} & \delta = 6, 7 \\ \{8\} & \delta = 8 \end{cases}$$

Proof. i) Let $m = 2$ and $\delta = 2$.Then a narrow sense binary BCH code with $\delta = 2$ is a cyclic code generated by $g(x) = x^2 + x + 1$. So $d_1 = 3$.

ii) Let $m = 3$. We will investigate the weight hierarchy for $\delta = 2, 3, 4$.
a) for $\delta = 2$ and $\delta = 3$, the polynomial $g(x) = x^3 + x + 1$ is the generator of this cyclic code. So $d_1 = 3$. We have $k = 4$. We will calculate d_2, d_3 and d_4. By theorem 2. 1, we have $d_2 \geq 4$. We will show that $d_2 = 4$. We know that

$$d_2 = min\{|X| : |X| - rank(< H_i > i \in X) = 2\}$$

where H is parity check matrix of code and H_i s are its columns.

Let $d_2 = 4$, so we should have 4 columns in H such that two of them are independent. Suppose H_t, H_s are these columns. The number of subspaces generated by these columns is
$\frac{2^2 - 1}{2 - 1} = \frac{3}{1} = 3$.

Then $d_2 \neq 4$. By Monotonicity, we have $5 \leq d_2 < d_3 < d_4 \leq 7$. Hence we have $d_2 = 5, d_3 = 6, d_4 = 7$.

b) If $\delta = 4$, by [5], we have $k = 7 - 6 = 1$. we have $g(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ as the generator of this cyclic code. Hence $d_1 = 7$.

iii) Let $m = 4$. We will investigate the result for several δ :

a) If $\delta = 2, 3$, the generator of this cyclic code is $1 + x + x^4$. So $d_1 = 3$. As we showed in previouse section, we have $d_2 = 5$.
Now we calculate d_3. Considering $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\}$ as an independent set of columns of parity check matrix H, we can add

$$\frac{2^3 - 1}{2 - 1} = \frac{8 - 1}{1} = 7$$

vectors to above set that cannot change the rank. On the other hand we have

$d_3 = min\{|X| : |X| - rank(< H_i >) = 3.$
Hence we have $d_3 = 6, d_4 = 7$ and we cannot use above set for d_5. Then $d_5 \geq 9$. By theorem 2. 1, we have $9 \leq d_5 < d_6 < d_7 < d_8 < d_9 < d_{10} < d_{11} \leq 15$.
Hence $d_5 = 9$ and $d_i = i + 4, 6 \leq i \leq 11$.

b) Let $\delta = 4$. By [5, p.162], we have $k = 7$. We computed the generator of this cyclic code and obtained it. $g(x) = x^8 + x^7 + x^6 + x^4 + 1$ is its generator. We have $d_1 = 5$. by [4], we have $d_2 = 8$. The structure of parity check matrix shows that $d^{\perp} = 4$. By [1], this code is 5-MDS code and it is not 3-MDS and 4-MDS. Then $d_4 \neq 12, d_3 \neq 11$. By Griesmer bound [6] and Generalized Singleton bound[1], we have
$d_3 \geqslant \lceil 5 \rceil + \lceil \frac{5}{2} \rceil + \lceil \frac{5}{4} \rceil = 5 + 3 + 2 = 10$ and $d_3 \leqslant 11$. Therefore $10 \leqslant d_3 \leqslant 11, d_3 \neq 11$, finally we have $d_3 = 10$. On the other hand, we have $d_4 \geqslant 11$ by theorem 2. 1, $d_4 \leqslant 12$(by generalized singleton bound[1]) and $d_4 \neq 12$. Therefore $d_4 = 11$. Since this code is 5-MDS, so $d_5 = 13$.

c) $\delta = 5$ is similar to $\delta = 4$.

d) If $\delta = 6$, then we have $g(x) = x^{10} + x^8 + x^5 + x^4 + x^2 + x + 1$ and $k = 5$. Hence We have $d_1 = 7$. Considering the structure of parity check matrix of this code, we have $d^{\perp} = 4$. Hence by [1], it is p_3-MDS code. Therefore $d_3 = 13$. Since it is not 2-MDS code, so $d_2 \neq 12$. By generalized singleton bound $d_2 \leqslant 12$ and by Griesmer bound $d_2 \geqslant 11$. Hence $d_2 = 11$.
e) If $\delta = 8$ we have $g(x) = x^{11} + x^{10} + x^9 + x^8 + x^6 + x^4 + x^3 + 1$. We have $k = 1$ and $d_1 = 8$.

□

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Some cocircuits of the splitting-off matroids

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Abstract

This paper describes the cocircuits of the splitting-off matroid M_{xy} in terms of the cocircuits of the original matroid M.

Keywords: Binary matroid, splitting-off, circuit, cocircuit, base.

Mathematics Subject Classification [2010]: 05B35

1 Introduction

Raghunathan et al. [3] extended the notation of splitting operation from graphs to binary matroids for every pair x, y of $E(M)$ and A. D. Mills [1] characterized some cocircuits of splitting matroids in term of cocircuits of original matroid.

Proposition 1.1. *[1] Let $M_{x,y}$ be the splitting matroid obtained from M so that $M \neq M_{x,y}$. Suppose $\{x, y\}$ is a proper subset of a cocircuit C^* of M. Then $\{x, y\}$ and $C^* - \{x, y\}$ are cocircuits of $M_{x,y}$.*

In this paper we describe the cocircuits of the splitting-off matroid M_{xy} in terms of the cocircuits of the original matroid M.

Let G be a graph. Given two adjacent non-loop edges $x = vv_1$ and $y = vv_2$ in G, we construct a new graph G_{xy} by adding the edge v_1v_2 and deleting the edges x and y. The transition from G to G_{xy} is called the splitting-off (or in short split-off) operation. We illustrate this construction with the help of Figure 1.

The split-off operation for a binary matroid with respect to a pair of elements is defined as follows [4]:

Definition 1.2. Let $M = (E; \mathcal{C})$ be a binary matroid on the set E together with the set $\mathcal{C}$ of circuits. Let $x, y \in E$ and "a" be the element that is not in E. We define:
$\mathcal{C}_0 = \{C \in \mathcal{C} : C$ contains neither x nor $y\}$;
$\mathcal{C}_1 = \{C - \{x, y\} \cup \{a\} : C \in \mathcal{C}$ and $x, y \in C\}$; and
$\mathcal{C}_2 = \{(C_1 \cup C_2 - \{x, y\}) \cup \{a\} : C_1, C_2 \in \mathcal{C}, C_1 \cap C_2 = \emptyset, x \in C_1, y \in C_2$ and $C_1 \cup C_2$ contains no circuit of M containing both x and y, or neither$\}$.

Further, we define $\mathcal{C}_{xy} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$ and $E' = ((E - \{x, y\}) \cup \{a\}$. Then the pair $((E - \{x, y\}) \cup \{a\}; \mathcal{C}_{xy})$ is a binary matroid M_{xy}. We call M_{xy} the split-off matroid.

[1] speaker

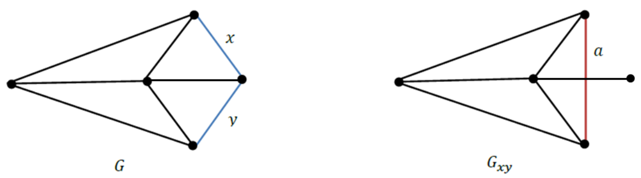

Figure 1:

Matrix representation of M_{xy}: Let M be a binary matroid on a set E, and let A be a matrix that represents M over $GF(2)$. Suppose that $x, y \in E$, and the element $a \notin E$. Let A_{xy} be the matrix obtained from A by adjoining an extra column, with label "a", which is the sum of the columns corresponding to x and y, and then deleting the two columns corresponding to x and y. Let M_{xy} be the vector matroid of the matrix A_{xy} over $GF(2)$. The transition from M to M_{xy} is called the split-off operation and the matroid M_{xy} is referred to as the split-off matroid.

Theorem 1.3. *[5] Let M be a binary matroid on a set E, $a \notin E$, and $x, y \in E$. Suppose that x, y are in series in M. Then B' is a base of M_{xy} if and only if one of the following two conditions holds.*
(i) *$B' = B - \{x\}$, where B is a base of M containing x but not y (respectively, $B' = B - \{y\}$, where B is a base of M containing y but not x).*
(ii) *$B' = (B - \{x, y\}) \cup \{a\}$, where B is a base of M containing both x and y.*

Definition 1.4. [2] The cocircuits of a matroid M are the minimal sets having non-empty intersection with every base of M.

The following propositions are necessary in our discussions.

Proposition 1.5. *[2] If C is a circuit and C^* is a cocircuit of a matroid. Then $|C \cap C^*| \neq 1$*

Proposition 1.6. *[2] The following statements are equivalent for a matroid M:*
(i) *M is binary.*
(ii) *If C is a circuit and C^* is a cocircuit, then $|C \cap C^*|$ is even.*
(iii) *If C_1 and C_2 are distinct circuits, then $C_1 \Delta C_2$ contains a circuit.*

Lemma 1.7. *[2] Let $\mathcal{X}$ and $\mathcal{Y}$ be collections of subsets of a finite set E such that every member of $\mathcal{X}$ contains a member of $\mathcal{Y}$, and every member of $\mathcal{Y}$ contains a member of $\mathcal{X}$. Then the minimal members of $\mathcal{X}$ are precisely the minimal members of $\mathcal{Y}$.*

In particular, the ground set and the collections of independent sets, bases, circuits and cocircuits of a matroid M will be denoted by $E(M)$, $\mathcal{I}(M)$, $\mathcal{B}(M)$, $\mathcal{C}(M)$ and $\mathcal{C}^*(M)$, respectively. The results in the next section describe cocircuits of M_{xy} in term of cocircuits M.

Various properties of splitting-off matroid are explored in [4] and [5]. For the standard terminology in matroid we refer to [2].

2 The cocircuits of the splitting-off matroids

It follows from definition 1.2 that if every circuit of M contains no member of x, y then $\mathcal{C}(M_{xy}) = \mathcal{C}(M)$. But $\mathcal{C}(M_{xy}) \neq \mathcal{C}(M)$ if there is a circuit of M containing both of x, y or there exist at least two disjoint circuits that x belongs to one of them and y to the other.

Proposition 2.1. *Let M be a binary matroid and x, y be two non-loop elements of M. Then $\{a\}$ is a cocircuit of M_{xy} if and only if $\{x\}$ or $\{y\}$ is a cocircuit of M.*

Proof. Suppose $\{x\}$ be a cocircuit of M but $\{y\}$ is not. Therefore, there isn't any circuit of M that contains x. So by circuits of M_{xy}, $\{a\}$ is a cocircuit of M_{xy}. By similar argument if $\{y\}$ be a cocircuit of M but $\{x\}$

is not, we conclude that $\{a\}$ is a cocircuit of M_{xy}. Now let $\{x\}$ and also $\{y\}$ are cocircuit of M. In this case by circuits of M_{xy}, $\{a\}$ is a cocircuit of M_{xy}.

Conversely, suppose that $\{a\}$ is the cocircuit of M. It is clear that x and y are in different components of M, otherwise there exist a circuit of M that contains x and y, a contradiction. If $\{x\}$ and $\{y\}$ are not cocircuits of M, then there are circuits C_1 and C_2 such that $x \in C_1$ and $y \in C_2$. If C_1 and C_2 are contained in different component of M, then $((C_1 \cup C_2) - \{x, y\}) \cup \{a\}$ is a circuit of M_{xy}, a contradiction. If C_1 and C_2 are contained in the same component of M, then there is a circuit C_3 of M containing x and y, therefore $(C_3 - \{x, y\}) \cup \{a\}$ will be a circuit of M_{xy}, which is a contradiction. So the result follows. □

Theorem 2.2. *Let M be a binary matroid on E, $x, y \in E$, and neither $\{x\}$ nor $\{y\}$ is a separator of M. If x and y are in series in M, then C^*_{xy} is a cocircuit of M_{xy} if and only if C^*_{xy} contained in one of the following sets:*
(i) *$\mathcal{C}_0 = \{C^*$, where C^* is a cocircuit of M such that $C^* \cap \{x, y\} = \emptyset\}$,*
(ii) *$\mathcal{C}_1 = \{(C^* - \{x, y\}) \cup \{a\}$, where C^* is a cocircuit of M containing exactly one of x or $y\}$.*

Proof. We first recall that since x and y are in series, then every base of M contains at least one element of $\{x, y\}$. Let $\{x, y\}$ is a circuit of M. Then $M = M_1 \oplus U_{1,2}$ for some matroid M_1. So we have $M_{xy} = M_1 \oplus U_{0,1}$. This completes the proof.

Suppose $\{x, y\}$ is not a circuit of M. Let $C^*_{xy} \in \mathcal{C}_0$. Then for some C^*, $C^*_{xy} = C^*$, where C^* is a cocircuit of M such that $C^* \cap \{x, y\} = \emptyset$. Since C^* intersects every base of M in elements different from x, y, so it intersects every base of M_{xy}. If C^*_{xy} is not minimal, then for some proper subset $D \subsetneqq C^*_{xy}$, D is a cocircuit of M_{xy}. By bases of M, M_{xy}, and the fact that D intersects every base of M_{xy} in some element different from x, y, we conclude that D intersects every base of M, this is a contradiction. Thus, C^*_{xy} is a cocircuit of M_{xy}.

Let $C^*_{xy} \in \mathcal{C}_1$. Then for some C^*, $C^*_{xy} = (C^* - \{x, y\}) \cup \{a\}$, where C^* is a cocircuit of M and contains exactly one of x or y. Without lose of generality, suppose $x \in C^*$ and $y \notin C^*$. We prove that C^*_{xy} intersects every base B' of M_{xy}.

If $B' = (B - \{x, y\}) \cup \{a\}$, then it is clear that $a \in C^*_{xy} \cap B'$. So we consider $B' = B - \{x\}$, where B is a base of M containing x but not y. Let $C^*_{xy} = (C^*_1 - \{x\}) \cup \{a\}$, for some cocircuit C^*_1 of M containing x but not y. Since C^*_1 and $\{x, y\}$ are cocircuits of M, then by proposition 1.6, $C^*_1 \Delta \{x, y\}$ is a cocircuit of M, so B contains at least one element of C^*_1. But $y \notin B$, so B contains at least one element of C^*_1 different from x. Thus, C^*_{xy} intersects the base B' of M_{xy}. By similar argument for any other base of M_{xy}, the condition holds. If C^*_{xy} is not minimal, then for some $D \subsetneqq C^*_{xy}$, D is a cocircuit of M_{xy}. If $a \notin D$, then by bases of M_{xy} and M, D also intersects every base of M. So D is a cocircuit of M that is contained in C^*, a contradiction. Let $a \in D$. In this case also by bases of M_{xy} and M, $(D - \{a\}) \cup \{x\}$ contains a cocircuit of M that is contained in C^*, a contradiction. So C^*_{xy} is a cocircuit of M_{xy}.

Conversely, suppose C^*_{xy} is a cocircuit of M_{xy}. We prove that C^*_{xy} contains some member of $\mathcal{C}_0$ or $\mathcal{C}_1$. Let $a \in C^*_{xy}$. If B is a base of M containing one element of $\{x, y\}$, then C^*_{xy} intersects B in some element different from x and y. So C^*_{xy} intersects every base of M containing one element of x and y. If C^*_{xy} intersects every base of M containing both x and y, in some element different from x and y, then $(C^*_{xy} - \{a\})$ is a cocircuit of M_{xy}, a contradiction. So there is some base B' of M_{xy} such that $a = B' \cap C^*_{xy}$, and $(B' - \{a\}) \cup \{x, y\}$ is a base of M. Hence $(C^*_{xy} - \{a\}) \cup \{x\}$ and $(C^*_{xy} - \{a\}) \cup \{y\}$ are cocircuits of M. Therefore C^*_{xy} contains some member of $\mathcal{C}_1$.

Let $a \notin C^*_{xy}$. Since x and y are in series in M, so every base of M contains at least one element of $\{x, y\}$. If B is a base of M with $x \in B$, and $y \notin B$, then $B' = B - \{x\}$ is a base of M_{xy}. But C^*_{xy} intersects B', so C^*_{xy} intersects B. By similar argument, if B is a base of M with $y \in B$ and $x \notin B$, we conclude C^*_{xy} intersects B.

Now let B is a base of M containing both x and y. Since C^*_{xy} intersects $B' = (B - \{x, y\}) \cup \{a\}$, so we conclude that C^*_{xy} meets B' in $B - \{x, y\}$. So C^*_{xy} intersects every base of M. This shows that C^*_{xy} contains some member of $\mathcal{C}_0$. Now by lemma 1.7, the result follows. □

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Super Connectivity Of a Family Of Direct Product Graphs

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Abstract

Let G_1 and G_2 be two graphs. The Kronecker product $G_1 \times G_2$ has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and the edge set $E(G_1 \times G_2) = \{(u_1, v_1), (u_2, v_2) \mid u_1u_2 \in E(G_1), v_1v_2 \in E(G_2)\}$. In this talk we show that if $K_{p_1,p_2,\cdots,p_r}$ is a complete multipartite graph, where the parameters $p_1, p_2, \cdots, p_r$ satisfying certain conditions and P_n is a path of length $n-1$, then $K_{p_1,p_2,\cdots,p_r} \times P_n$ is not super-$k^{(i)}$, where $1 \leq i \leq n-1$ and $n \geq 2$. Also we show that $K_{p_1,p_2,\cdots,p_r} \times C_n$ is not super-$k^{'}$, where C_n is a cycle of length n and $n \geq 3$.

Keywords: direct product, super connectivity, vertex-cut.
Mathematics Subject Classification [2010]: Primary: 22D15, 43A10; Secondary: 43A20, 46H25

1 Introduction

Let $G = (V(G), E(G))$ be a graph where $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G. For two vertices $u, v \in V(G)$, u and v are *neighbors* if u and v are adjacent, that is, if there is an edge $e = uv$. Also we write $u \sim v$. The neighborhood of a vertex v in G, denoted by $N_G(v)$ (or $N(v)$), is $\{u \mid uv \in E(G)\}$.

The *degree* of v, denoted by $d_G(v)$, is equal to the number of neighbors of v, that is $d_G(v) = |N_G(v)|$. The number $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$ is the minimum degree of G. For a subset $S \subseteq V(G)$ the neighborhood of S is $N(S) = \cup_{v \in S} N(v)$. The subgraph induced by S is denoted by $G[S]$. As usual $K_{p_1,p_2,\cdots,p_r}$ denotes the complete multipartite graph and P_n is the path of length $n-1$.

The notion of super-connectedness proposed in [1, 2, 3] aims at pushing the analysis of the connectivity properties of graphs beyond the standard connectivity. A graph G is *super connected*, or simply *super-κ*, if every minimum separating set is the neighbors of a vertex of G, that is, every minimum separating set isolates a vertex. Observe that a super-connected graph G is necessarily maximally connected, i.e., $\kappa(G) = \delta(G)$, but the converse is not true. It is easy to see from the cycle graph C_n ($n \geq 6$). Also a graph G is *super m-connected*, or simply *super-$\kappa^{(m)}$*, if every minimum separating set isolates a component of size m. Also we should note that for $m = 1$ super-$\kappa^{(1)}$ and super-$\kappa^{'}$ have same concept.

Miller [9] and Weichsel [12] investigated the connectedness of Kronecker product of two connected graphs. Guji and Vumer [5] presented the connectivity of Kronecker product of a bipartite graph and a complete graph. For more results on the connectivity of Kronecker products we refer the reader to [4, 6, 8, 10].

[1]speaker

Wang et al. [10] investigeted the super-connectivity of the Kronecker product of a non-bipartite graph and a complete graph K_n $(n \geq 3)$. Also Gua et al. [7] proved that $K_m \times K_n$ is super-κ for $n \geq m \geq 2$ and $n + m \geq 5$, $K_m \times P_n$ is super-κ for $n \geq m \geq 3$.

2 Main Results

Suppose that $G = K_{p_1,p_2,\cdots,p_r} \times H$, where H is an arbitrary graph of order n. Also suppose that $K_{p_1,p_2,\cdots,p_r}$ is a complete multipartite graph with partitions $(W_1, W_2, \cdots, W_r)$. We let $|V(W_i)| = p_i$ and $S_i = V(K_{p_1,p_2,\cdots,p_r}) \times \{v_i\}$, where $v_i \in V(H)$ and $1 \leq i \leq n$. Moreover for convenience we shall abbreviate (v_i, v_j) as w_{ij} for $v_i \in V(K_{p_1,p_2,\cdots,p_r})$ and $v_j \in V(H)$. Also we say that v_i is the first component and v_j is the second component. It is easy to see that $V(K_{p_1,p_2,\cdots,p_r} \times H) = S_1 \cup S_2 \cup \cdots \cup S_n$. Also S_i is called the layer of $K_{p_1,p_2,\cdots,p_r} \times H$. If $i > t$ then we say that S_i is a layer upper than S_t and S_t is a layer lower than S_i.

Suppose that $K_{p_1,p_2,\cdots,p_r}$ is the complete multipartite graph which satisfies the following conditions:
1) $r \geq 3$;
2) $p_1 \leq p_2 \leq \cdots \leq p_r$;
3) $\sum_{i=1}^{r-2} p_i \geq p_{r-1}$;
4) $\sum_{i=1}^{r-1} p_i \geq p_r$.

Lemma 2.1. *Suppose that $G = K_{p_1,p_2,\cdots,p_r} \times P_n$ and S is a vertex-cut with $|S| = \sum_{i=1}^{r-1} p_i$. Also suppose that C is an arbitrary component of $G - S$ such that $|C| \geq 2$. Then each S_i $(1 \leq i \leq n)$ has at least one vertex in C.*

Proof. Suppose to contrary that there exists a S_i such that $V(S_i) \cap V(C) = \emptyset$. It is easy to see that there are at least two layers, say S_l and S_k, which have intersection with C. Let $w_{al} \in V(S_l) \cap V(C)$ and $w_{bk} \in V(S_k) \cap V(C)$. Without loss of generality we may suppose that $l > k$. If S_l is upper than S_i and S_k is lower than we get a contradiction. Thus we may suppose that both of S_l and S_k either are upper than S_i or lower than S_i. Again we get a contradiction in this case.

Suppose that $G = K_{p_1,p_2,\cdots,p_r} \times P_n$. Considering the conditions $(1) - (4)$ for $p_1, p_2, \cdots, p_r$. By [11, Theorem 1.4], $\kappa(K_{p_1,p_2,\cdots,p_r} \times H) = \min\{\sum_{i=1}^{r} p_i\kappa(H), \sum_{i=1}^{r-1} p_i\delta(H)\}$, where H is an arbitrary graph. Thus $\kappa(K_{p_1,p_2,\cdots,p_r} \times P_n) = \sum_{i=1}^{r-1} p_i$.

In the following theorem we show that every minimum vertex cut of $K_{p_1,p_2,\cdots,p_r} \times P_2$ can not isolate a vertex.

Lemma 2.2. *Suppose that $p_1, p_2, \cdots, p_r$ satisfies conditions $(1) - (4)$ as above and $G = K_{p_1,p_2,\cdots,p_r} \times P_2$. Then G is not super-κ'.*

Theorem 2.3. *Suppose that $p_1, p_2, \cdots, p_r$ satisfies conditions $(1) - (4)$ as above and $G = K_{p_1,p_2,\cdots,p_r} \times P_n$, where $n \geq 3$. Then G is not super-$\kappa^{(i)}$, where $1 \leq i \leq n - 1$.*

Proof. Suppose to contrary that G is super-$\kappa^{(i)}$, where $1 \leq i \leq n - 1$. First suppose that $i = 1$. Thus for every minimum vertex cut, say S, isolated vertex. We have $|S| = \sum_{i=1}^{r-1} p_i$. Now we can get a contradiction. Thus $S_{k-1} \nsubseteq S$ and $S_{k+1} \nsubseteq S$ and both of S_{k-1} and S_{k+1} have vertex in $G - S$. Now if the first component of this vertex is in different partition from W_j then we get a contradiction

Now suppose that $2 \leq i \leq n - 1$. Again in this case we get a contradiction.

Theorem 2.4. *Suppose that $p_1, p_2, \cdots, p_r$ satisfies conditions $(1), (3)$ and (4). Also suppose that $p_1 < p_2 < \cdots < p_r$ and $G = K_{p_1,p_2,\cdots,p_r} \times C_n$, where $n \geq 3$. Then G is not super-κ'.*

Proof. Suppose to contrary that G is super-κ. Thus for every minimum vertex cut, say S, isolated vertex. Let the isolated vertex be w_{lk} and C be a component which contain this isolated vertex. Also let W_j be a partition which contain v_l and $|W_j| = p_j$ Now we consider the following cases and we get a contradiction in each case.
Case 1. $S_{k-1}, S_{k+1} \subseteq S$.
Case 2. Either $S_{k-1} \subseteq S$ or $S_{k+1} \subseteq S$.

Now we consider the following subcase and get a contradiction in each subcases.
Subcase 2.1. $j = r$.
Subcase 2.2. $j \neq r$.
Case 3. $S_{k-1}, S_{k+1} \nsubseteq S$.
Now we consider the following subcases and get a contradiction in each subcases.
Subcase 3.1. $j \neq r$.
Subcase 3.2. $j = r$.

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Characterization of ternary Spikes

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Abstract

Spikes are an important class of 3-connected matroids. For an integer $r \geq 3$, each matroid that is obtained by relaxing one of the circuit-hyperplanes of an r-spike with rank r is isomorphic to another r-spike and repeating this procedure will produce other r-spikes. For all $r \geq 3$, there are exactly two ternary spikes. In this paper, We give an algorithm to characterize the structure of these ternary spikes.

Keywords: Matroid, Circuit-hyperplane, Relaxing, Spike.
Mathematics Subject Classification [2010]: 05B35

1 Introduction

The matroid notation and terminology used here will follow Oxley [2]. Let $E = \{x_1, x_2, ..., x_r, y_1, y_2, ..., y_r, t\}$ for some $r \geq 3$. Let $\mathcal{C}_1 = \{\{t, x_i, y_i\} : 1 \leq i \leq r\}$ and $\mathcal{C}_2 = \{\{x_i, y_i, x_j, y_j : 1 \leq i < j \leq r\}$. The set of circuits of every spike on E includes $\mathcal{C}_1 \cup \mathcal{C}_2$. Let $\mathcal{C}_3$ be a, possibly empty, subsets of $\{\{z_1, z_2, ..., z_r\} : z_i$ is in $\{x_i, y_i\}$ for all $i\}$ such that no two members of $\mathcal{C}_3$ have more than $r-2$ common elements. Finally, let $\mathcal{C}_4$ be the collection of all $(r+1)$-element subsets of E that contain no member of $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. The matroid M whose collection $\mathcal{C}$ of circuits is $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ is called a *rank-r spike with tip t* and *legs* $L_1, L_2, ..., L_r$ where $L_i = \{t, x_i, y_i\}$ for all i. In an arbitrary spike M, each circuit in $\mathcal{C}_3$ is also a hyperplane of M and when one of the circuit-hyperplanes is relaxed, we obtain another spike. If $\mathcal{C}_3$ is empty, the corresponding spike is called the rank-r free spike with tip t.

Let E be a ground set of a $GF(p)$-representable spike M. We wish to determine what subsets of E are members of $\mathcal{C}_3$ and how can we recognize a matrix that represents M. This leads to computing the number of circuit-hyperplanes of M and producing many of spikes which are not $GF(p)$-representable, from M by relaxing operation.

Let $\mathbb{F}$ be a field and $\alpha_1, \alpha_2, ..., \alpha_r$ be non-zero elements of $\mathbb{F}$. Let $\mathbf{1}$ be the $r \times 1$ matrix of all ones, and let

$$A_r = \begin{array}{c} \begin{matrix} y_1 & y_2 & y_3 & \cdots & y_{r-1} & y_r \end{matrix} \\ \begin{bmatrix} 1+{\alpha_1}^{-1} & 1 & 1 & \dots & 1 & 1 \\ 1 & 1+{\alpha_2}^{-1} & 1 & \dots & 1 & 1 \\ 1 & 1 & 1+{\alpha_3}^{-1} & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1+{\alpha_{r-1}}^{-1} & 1 \\ 1 & 1 & 1 & \dots & 1 & 1+{\alpha_r}^{-1} \end{bmatrix} \end{array}. \quad (1)$$

Geelen, Gerards, and Whittle (2002) [1] have described one key result for the representability of spikes as follows.

[1]Speaker

Proposition 1.1. *Suppose that $r \geq 3$ and $\mathbb{F}$ is a field. Let M be an $\mathbb{F}$-representable rank-r spike with legs $\{t, x_1, y_1\}, \{t, x_2, y_2\}, ..., \{t, x_r, y_r\}$ such that $\{x_1, x_2, ..., x_r\}$ is independent. Then every $\mathbb{F}$-representation of M is projectively equivalent to a matrix of the form $[I_r|A_r|1]$ whose columns are labeled, in order, $x_1, x_2, ..., x_r, y_1, y_2, ..., y_r, t$, where A_r is as in (1). Moreover, for $K \subseteq \{1, 2, ..., r\}$, the set $\{x_k : k \notin K\} \cup \{y_k : k \in K\}$ is a circuit of M if and only if $\sum_{k \in K} \alpha_k = -1$.*

The following result of Oxley and Whittle (1998) [4] gives a necessary and sufficient condition for a matroid that is obtained by relaxing operation to be ternary.

Proposition 1.2. *Let M be a matroid that is obtained by relaxing a circuit-hyperplane H in a ternary matroid N. Then M is ternary if and only if N is isomorphic to the cycle matroid of a graph that is obtained from a cycle with edge set H by adjoining an extra vertex v and then adding some non-zero number of edges from v to the vertices of H.*

2 Main Results

In this section, we introduce one way to find all members of $\mathcal{C}_3 \cup \mathcal{C}_4$ for a ternary spike M with the matrix representation $[I_r|A_r|1]$ whose columns are labeled, in order, $x_1, x_2, ..., x_r, y_1, y_2, ..., y_r, t$. Since there is no graphic spike, by Proposition 1.1, we can see the construction of all non-ternary spikes that are obtained from M by relaxing operation. For this section, we let X and Y be disjoint r-element sets $\{x_1, x_2, ..., x_r\}$ and $\{y_1, y_2, ..., y_r\}$, respectively. We also let i in $\{1, 2, ..., r\}$, j in $\{1, 2\}$, and for p prime, two integers k and k' are in $\{1, 2, ..., p\}$.

Definition 2.1. Let $\mathbb{F}$ be a prime field $GF(p)$ and r be an integer exceeding two. For each $n, k \in \mathbb{N}$, with $k \leq p$ we define

$$\mathcal{P}_{n_k} = \{pn - k : n \in \mathbb{N}\}. \tag{2}$$

For all distinct k and k' in $\mathbb{N}$, we have $\mathcal{P}_{n_k} \cap \mathcal{P}_{n_{k'}} = \emptyset$. Let $n \leq \frac{k+r}{p}$. Then $\bigcup_{k=1}^{p} \mathcal{P}_{n_k} = \{0, 1, ..., r\}$ and so $(\mathcal{P}_{n_1}, \mathcal{P}_{n_2}, ..., \mathcal{P}_{n_p})$ is a partition of $\{0, 1, ..., r\}$. We say that such partition is a *$GF(p)$-partition of r*.

Definition 2.2. Let $\mathcal{P}_{n_k}$ be the set defined in Definition 2.1. For all primes p, the field $GF(p)$ coincides with $\mathbb{Z}_p$, the ring of integers modulo p. Thus all members of $\mathcal{P}_{n_k}$ are congruent to $p - k$ modulo p. We denote this value by $tr_p(\mathcal{P}_{n_k})$ and call it *$GF(p)$-trace of $\mathcal{P}_{n_k}$*.

Let M_r be an r-spike with respect to A_r over $GF(3)$. Then $E(M) = X \cup Y \cup t$. Let $B = (\beta_1, \beta_2, ..., \beta_r)$ be the ordered r-tuples of diagonal entries of A_r such that β_i is a corresponding entry in the ith row and column of A_r. Then, for all i, we have $\beta_i = 1 + \alpha_i^{-1}$, so $\beta_i \in \{0, -1\}$. We denote by $\mathcal{D}(A_r)$ the set of non-zero elements of B.

Let (Y_1, Y_2) be a partition of Y such that if $\beta_i = 0$, then $y_i \in Y_1$, otherwise $y_i \in Y_2$ where Y_1 or Y_2 may be empty. This partition is unique and we call it *$GF(3)$-partition of Y*. Now the $GF(3)$-partition of r is $(\mathcal{P}_{n_1}, \mathcal{P}_{n_2}, \mathcal{P}_{n_3})$ where

$$\mathcal{P}_{n_k} = \{3n - k : n \in \mathbb{N} \text{ and } n \leq \frac{k+r}{3}\}. \tag{3}$$

For an r-spike M_r with respect to A_r over $GF(p)$, we denote by *$\ddot{r}$-element set* the subset of $E(M_r)$ that has cardinality r and contains at most one element of each $\{x_i, y_i\}$ for all i.

Theorem 2.3. *Let M_r be an r-spike with respect to A_r over $GF(3)$. Let (Y_1, Y_2) be the $GF(3)$-partition of Y. An $\ddot{r}$-element set is a circuit-hyperplane of M_r if and only if it is a member of one of the following sets:*

(i) $\{Z : |Z \cap Y_1| \in \mathcal{P}_{n_3}$ *and* $|Z \cap Y_2| \in \mathcal{P}_{n_1}\}$;

(ii) $\{Z : |Z \cap Y_1| \in \mathcal{P}_{n_2}$ *and* $|Z \cap Y_2| \in \mathcal{P}_{n_3}\}$; *or*

(iii) $\{Z : |Z \cap Y_1| \in \mathcal{P}_{n_1}$ *and* $|Z \cap Y_2| \in \mathcal{P}_{n_2}\}$.

Note that, if $|\mathcal{D}(A_r)| = r$, then $Y_1 = \emptyset$ and so the sets of all circuit-hyperplanes of M_r is $\{Z : |Z \cap Y_2| \in \mathcal{P}_{n_1}\}$, and if $|\mathcal{D}(A_r)| = 0$, then $Y_2 = \emptyset$ and therefore the sets of all circuit-hyperplanes of M_r is $\{Z : |Z \cap Y_1| \in \mathcal{P}_{n_2}\}$.

Corollary 2.4. *Let M_r be an r-spike with respect to A_r over $GF(3)$. Let $|\mathcal{D}(A_r)| = m$. Then the number of circuit-hyperplanes of M is*

$$\sum_{l\in\mathcal{P}_{n_3}} \sum_{n\in\mathcal{P}_{n_1}} \tbinom{m}{n}\tbinom{r-m}{l} + \sum_{l\in\mathcal{P}_{n_2}} \sum_{n\in\mathcal{P}_{n_3}} \tbinom{m}{n}\tbinom{r-m}{l} + \sum_{l\in\mathcal{P}_{n_1}} \sum_{n\in\mathcal{P}_{n_2}} \tbinom{m}{n}\tbinom{r-m}{l}. \tag{4}$$

Theorem 2.5. *Let M_r be an r-spike with respect to A_r over $GF(3)$. Let (Y_1, Y_2) be the $GF(3)$-partition of Y. A union of an $\ddot{r}$-element set which is not a circuit-hyperplane and one element of $E(M_r)$ is an $(r+1)$-circuit of M_r if and only if it is a member of one of the following sets:*

- **(i)** $\{Z \cup z : \; |Z \cap Y_j| \in \mathcal{P}_{n_k}$ *for all* j *in* $\{1, 2\}$ *and* $z \in ((Y_2 - Z) \cup t)\}$*; or*
- **(ii)** $\{Z \cup z : |Z \cap Y_1| \in \mathcal{P}_{n_k}$ *and* $|Z \cap Y_2| \in \mathcal{P}_{n_{k'}}, k \neq k'$ *and* $z \in ((Y_1 - Z) \cup t)\}$.

Note that, if $|\mathcal{D}(A_r)| = r$, then $Y_1 = \emptyset$ and so the sets of all $(r+1)$-circuit of M_r is the union of the sets consists of

- **(i)** $\{Z \cup z : |Z \cap Y_2| \in \mathcal{P}_{n_3}$ and $z \in ((Y_2 - Z) \cup t\}$;
- **(ii)** $\{Z \cup t : |Z \cap Y_2| \in \mathcal{P}_{n_2}\} \cup \{Z \cup t : |Z \cap Y_2| \in \mathcal{P}_{n_1}\}$.

Moreover, if $|\mathcal{D}(A_r)| = 0$, then $Y_2 = \emptyset$ and therefore the sets of all $(r+1)$-circuits of M_r is the union of the sets obtained from (i)-(ii) listed as above by interchanging Y_2 and Y_1.

Corollary 2.6. *Let M_r be an r-spike with respect to A_r over $GF(3)$. Let $|\mathcal{D}(A_r)| = m$ and (k, k') be a pair of numbers such that $(k, k') \notin \{(3, 1), (2, 3), (1, 2)\}$. Then the number of $(r+1)$-circuits of M is*

$$\begin{aligned}\sum_{k=1}^{3} \Big(\sum_{l\in\mathcal{P}_{n_k}} \sum_{n\in\mathcal{P}_{n_k}} \binom{m}{n}\binom{r-m}{l}(m-n+1)\Big) + \\ \sum_{k\neq k'} \Big(\sum_{l\in\mathcal{P}_{n_k}} \sum_{n\in\mathcal{P}_{n_{k'}}} \tbinom{m}{n}\tbinom{r-m}{l}(r-m-l+1)\Big).\end{aligned} \tag{5}$$

By the fact that there are exactly two ternary r-spike, any of the summation formulas (4) and (5) has exactly two different values. For example, if $|\mathcal{D}(A_r)| = r$ and $|\mathcal{D}(A_r)| = 1$ we get these to different values.

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Normal subgroup graph of a group

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Abstract

Given any normal subgroup N of a group G, we let $\Gamma_N(G)$ be the graph with vertex set

$$\{x \in G \setminus N \mid xy \in N \text{ for some } y \in G \setminus N\}$$

, where distinct vertices x and y are adjacent if and only if $xy \in N$. In this article, we show that, $\Gamma_N(G)$ is connected with $diam(\Gamma_N(G)) \leq 3$. Furthermore, if $\Gamma_N(G)$ contains a cycle, then $gr(\Gamma_N(G)) \leq 7$. Also, we show that, if N is a nonzero proper normal subgroup of a group G, then $\Gamma_N(G)$ has no cut-points.

Keywords: Graph, Group, Normal subgroup.

Mathematics Subject Classification [2010]: Primary: 05C25; Secondary: 20B05

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1 Introduction

In 1988, Istvan Beck [4] opened up the fascinating insight which relates a graph with the algebraic structure ring. He introduced the zero divisor graph of a commutative ring, and later on, this introduction was slightly modified by Anderson and Naseer in [1]. Many authors studied the zero divisor graph in the sense of Anderson-Livingston as in [2]. Recently, there has been considerable attention to associating graphs with algebraic structures. Much of this has been motivated by the zero-divisor graph $\Gamma(R)$ of a commutative ring R as introduced in [3].

In the following, we recall some definitions, notations, and results concerning elementary graph theory. By a graph $G = (V, E)$, the authors mean a nonempty set V and a symmetric binary relation (possibly empty) E on V. The set V is called the set of vertices and E is called the set of edges of G. Two elements x and y in V are said to be adjacent if $(x, y) \in E$ and we write $x \sim y$. For a graph G, let $d(x, y)$ be the length of the shortest path from x to a distinct y in G (and let $d(x, y) = \infty$ if no such path exists). The diameter of G is zero if G is the graph on one vertex and is $diam(G) = sup\{d(x, y) \mid x, y \text{ are distinct vertices of } G\}$ otherwise. A cycle in a graph G is a path that begins and ends at the same vertex. The girth of G, written $gr(G)$, is the length of the shortest cycle in $gr(G)$ (and $gr(G) = \infty$ if G has no cycles) [5, 7].

In this article, let G be a group and let N be a normal subgroup of G such that $xN = N$ for every $x \in G \setminus N$. We define an undirected graph $\Gamma_N(G)$ with vertices $\{x \in G \setminus N \mid xy \in N \text{ for some } y \in G \setminus N\}$, where distinct vertices x and y are adjacent if and only if $xy \in N$. we shwo that, $\Gamma_N(G)$ is connected with $diam(\Gamma_N(G)) \leq 3$. Furthermore, if $\Gamma_N(G)$ contains a cycle, then $gr(\Gamma_N(G)) \leq 7$ (Theorem 2.2). Finally, we show that, if N is a nonzero proper normal subgroup of a group G, then $\Gamma_N(G)$ has no cut-points (Theorem 2.6).

[1]speaker

2 Main Results

In this article, we assume that if N is a normal subgroup of G, then $xN = N$ for every $x \in G \setminus N$.

Definition 2.1. Let G be a group and N be a normal subgroup of G. We define a graph $\Gamma_N(G)$ with vertices $\{x \in G \setminus N \mid xy \in N \text{ for some } y \in G \setminus N\}$, where distinct vertices x and y are adjacent if and only if $xy \in N$.

Theorem 2.2. *Let N be a normal subgroup of a group G. Then $\Gamma_N(G)$ is connected with $diam(\Gamma_N(G)) \leq 3$. Furthermore, if $\Gamma_N(G)$ contains a cycle, then $gr(\Gamma_N(G)) \leq 7$.*

Proof. Let x and y be distinct vertices of $\Gamma_N(G)$. Then

Case 1. Let $xy \in N$. Then $x \sim y$ is a path in $\Gamma_N(G)$.

Case 2. Let $xt \notin N$, $x^2 \in N$ and $y^2 \in N$. Then $x \sim xy \sim y$ is a path.

Case 3. Let $xy \notin N$, $x^2 \in N$ and $y^2 \notin N$. Then there is some $b \in G \setminus N$ such that $by \in N$. If $bx \in N$, then $x \sim b \sim y$ is a path. If $bx \notin N$, then $x \sim bx \sim y$ is a path.

Case 4. Let $xy \notin N$, $y^2 \in N$ and $x^2 \notin N$. Then we obtain a path as in the above case.

Case 5. Let $xy \notin N$, $x^2 \notin N$ and $y^2 \notin N$. Then there exist $a, b \in R \setminus (N \cup \{x, y\})$ such that $ax \in N$ and $by \in N$. If $a = b$, then $x \sim a \sim y$ is a path. If $a \neq b$ and $ab \in N$, then $x \sim a \sim b \sim y$ is a path. If $b \neq b$ and $ab \neq N$, then $x \sim ab \sim y$ is a path. Thus $\Gamma_N(G)$ is connected and $diam(\Gamma_N(G)) \leq 3$. For any undirected graph G, $gr(G) \leq 2diam(G) + 1$, if G contains a cycle by [6, Proposition 1.3.2]. Thus $gr(\Gamma_N(G)) \leq 7$. □

Theorem 2.3. *Let N be a normal subgroup of a group R, and let $x, y \in G \setminus N$. Then the following statements hold.*

(1) If $x + I$ is adjacent to $y + I$ in $\Gamma_N(\frac{G}{N})$, then x is adjacent to y in $\Gamma_N(G)$.

(2) If x is adjacent to y in $\Gamma_N(G)$ and $x + N \neq y + N$, then $x + I$ is adjacent to $y + I$ in $\Gamma_N(\frac{G}{N})$.

(3) If x is adjacent to y in $\Gamma_N(G)$ and $x + N = y + N$, then $x^2, y^2 \in N$.

Proof. Clear. □

Corollary 2.4. *If x and y are (distinct) adjacent vertices in $\Gamma_N(G)$, then all (distinct) elements of $x + I$ and $y + I$ are adjacent in $\Gamma_N(G)$. If $x^2 \in I$, then all the distinct elements of $x + I$ are adjacent in $\Gamma_N(G)$.*

Definition 2.5. A vertex x of a connected graph G is a cut-point of G if there are vertices u, w of G such that x is in every path from u to w (and $x \neq u$, $x \neq w$). Equivalently, for a connected graph G, x is a cut-point of G if $G \setminus \{x\}$ is not connected.

Theorem 2.6. *If N is a nonzero proper normal subgroup of a group G, then $\Gamma_N(G)$ has no cut-points.*

Proof. Assume the vertex x of $\Gamma_N(G)$ is a cut-point. Then there exist vertices $u, w \in G \setminus N$ such that x lies on every path from u to w. By Theorem 2.2, the shortest path from u to w is of length 2 or 3.

Case 1. Suppose $u \sim x \sim w$ is a path of shortest length from u to w. If $x + N = u + N$, then x adjacent w implies u is adjacent to w by Corollary 2.4. Similarly, if $x + N = w + N$, u is adjacent to w. So suppose $x + N \neq u + N$ and $x + N \neq w + N$. Let $0 \neq n \in N$. Then $ux, xw \in N$ imply $u(x + n), w(x + n) \in N$. Hence $u \sim (x + n) \sim w$ is a path in $\Gamma_N(G)$. Thus in all cases we get a contradiction.

Case 2. Suppose (without loss of generality) $u \sim x \sim y \sim w$ is a path of shortest length from u to w in $\Gamma_N(G)$. If $x + N = y + N$, then u adjacent to x implies u is adjacent to y and therefore $u \sim y \sim w$ is a path. If $x + N \neq y + N$ and $0 \neq n \in N$, then as above, u and y adjacent to x means that u and y are also adjacent to $x + n$. Hence $u \sim (x + n) \sim y \sim w$ is a path. Thus in all cases we get a contradiction. □

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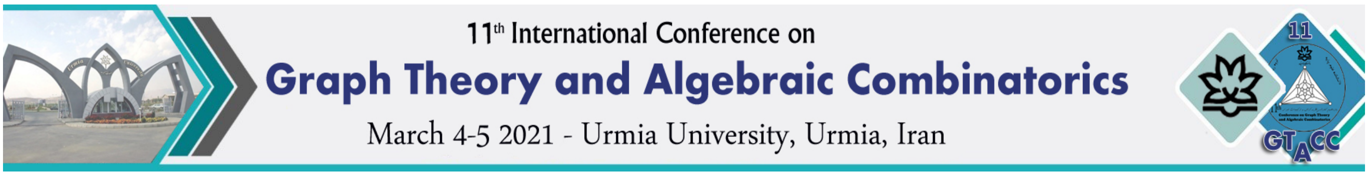

Some Inequalities with Respect to The Difference Between Atom-bond Connectivity Index and Randić Index

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Abstract

Let G be a simple and connected graph with vertex set $V(G)$ and edge set $E(G)$. The atom-bond connectivity and Randić index of graph G are two well-defined topological indices in chemical graph theory. Recently, Ali and Du [On the difference between atom-bond connectivity index and Randić index of binary and chemical trees, Int. J. Quantum Chem. (2017) e25446] characterized some extremal results with the respect to the difference between atom-bond connectivity index and Randić index for n-vertex binary and chemical trees. In this paper, we establish some inequalities with respect to the difference between atom-bond connectivity index and Randić index and study the relation between them.

Keywords: Atom-bond connectivity index, Randić index, Difference between topological indices, Graph.

Mathematics Subject Classification [2010]: Primary: 05C35; Secondary: 05C90

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and and edge set $E(G)$. Let d_u denote the degree of vertex $u \in E(G)$. Let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of G, respectively. For $x, y \in V(G)$, then the distance $d_G(x, y)$ between x and y is defined as the length of any shortest path in G connecting x and y. For a vertex v_i of $V(G)$, its eccentricity e_i is the largest distance between v_i and any other vertex v_k of G, i.e, $e_i = \max_{v_j \in V(G)} d_G(v_i, v_j)$. For the notations and terminologies in Graph Theory which are not mentioned here, please refer to [25].

Graph theory has provided the chemists with a variety of useful tools, one of which is the topological indices [7]. Molecules and molecular compounds are often modeled by molecular graphs. Topological indices of molecular graphs are one of the oldest and the most widely used descriptors in QSPR/QSAR research [22].

In 1975, Randić [17] proposed the Randić index for measuring the extent of branching of the carbon-atom skeleton of saturated hydrocarbon, and this index was defined as follows:

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$

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Estrada *et al.* [10] proposed a topological index named as atom-bond connectivity (ABC for short) index using a modification of Randić index. The ABC index of G is defined as

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}.$$

This index became popular only ten years later, when the paper [9] was published. The mathematical properties of the ABC index have been studied extensively. For the details, see the surveys [13], the recent papers [4, 8, 12, 21, 26] and the references cited therein.

In line with the great interest of topological indices, some researchers are also keen on the research of the relationship or comparison between topological indices, for example see [5, 6, 19, 28]. Very recently, Ali and Du [1] established some extremal results with the respect to the difference between ABC index and Randić index (R index for short) for binary and chemical trees. A tree T is said to be a binary tree (chemical tree, respectively) if the maximum vertex degree in T is at most 3 (4, respectively).

For $n \geq 3$, if G is an n-vertex connected graph, then

$$(ABC - R)(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u + d_v - 2} - 1}{\sqrt{d_u d_v}}.$$

It can be easily seen that $(ABC - R)(G) \geq 0$ with equality if and only if G is isomorphic to P_3, the 3-vertex path graph. For the remaining part of this article, we take $n \geq 4$.

The first Zagreb index can also be expressed as a sum over edges of G [14],

$$M_1(G) = \sum_{uv \in E(G)} (d_u + d_v).$$

Further results on Zagreb indices please refer [11, 27, 24], recent surveys [2, 3] and the references cited therein.

The modified second Zagreb index $\mathcal{M}_2^*(G)$ is equal to the sum of the reciprocal products of degrees of pairs of adjacent vertices [20], that is,

$$\mathcal{M}_2^*(G) = \sum_{uv \in E(G)} \frac{1}{d_u d_v}.$$

We make use of the following inequalities in this paper.

Theorem 1.1. *(Jensen's inequality [15]) Let $a = (a_i)_{i=1}^n$ and $p = (p_i)_{i=1}^n$, be two sequences of positive real numbers. Then, for any real number r with $r \leqslant 0$ or $r \geqslant 1$,*

$$\sum_{i=1}^n p_i a_i^r \geqslant \sum_{i=1}^n p_i \left(\frac{\sum_{i=1}^n p_i a_i}{\sum_{i=1}^n p_i} \right)^r. \tag{1}$$

Theorem 1.2. *([18]) Let $a = (a_i)$ and $(b_i), i = 1, 2, \ldots, n$, be two sequences of positive real numbers. For any $r \geqslant 0$,*

$$\sum_{i=1}^n \frac{a_i^{r+1}}{b^r} \geqslant \frac{\left(\sum_{i=1}^n a_i\right)^{r+1}}{\left(\sum_{i=1}^n b_i\right)^r}. \tag{2}$$

In this paper, we establish some upper and lower bounds for this new graph invariant, and study the relation between them.

2 Main Results

In this section, we only present two results for upper and lower bounds of $ABC - R$ index.

Theorem 2.1. *Let G be a graph with m edges, minimum degree δ and the first Zagreb index $M_1(G)$. Then*

$$(ABC - R)(G) < \sqrt{(M_1(G) - m)\frac{m}{\delta^2}}.$$

Theorem 2.2. *Let G be a graph with m edges, minimum degree δ, maximum degree Δ and modified second Zagreb index $\mathcal{M}_2^*(G)$. Then*

$$\left(\sqrt{2\delta - 2} - 1\right)\sqrt{\mathcal{M}_2^*(G) + \frac{m(m-1)}{\Delta^2}} \leqslant (ABC - R)(G) \leqslant \left(\sqrt{2\Delta - 2} - 1\right)\sqrt{\mathcal{M}_2^*(G) + \frac{m(m-1)}{\delta^2}}$$

with equality if and only if G is a regular graph.

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Sums of Polynomials and Clique Roots

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Abstract

In this paper, we first introduce a key lemma which indicates a necessary condition for having a real root for sums of polynomials with (at least) one real root. Then, as an immediate application of this simple but potentially useful lemma we characterize several classes of graphs which have only clique roots. Finally, we conclude our paper with several interesting open problems and conjectures for interested readers.

Keywords: Sums of polynomials, Clique polynomials, Clique roots, Chordal graphs.
Mathematics Subject Classification [2010]: Primary: 05C31, 05C69; Secondary: 30C15.

1 Introduction

The property of having only real roots for graph polynomials of some special classes of graphs is of special interest for algebraic graph theorists in recent years. The reason is that having only real roots for a graph polynomial results in detecting many combinatorial features for the given graph.
One of the most interesting breaking through in mathematical sciences in recent years was the solution of the well-known *Kadsion-Singer* conjecture which was open for more than 50 years. The main idea behind the proof was introducing an interlacing family of polynomials (see [1] and reference therein for more details).
We are not really aware of the origin of interlacing idea, but it seems that one of the early motivations comes from the famous *Rolle*'s theorem in the calculus of one-variable functions. An immediate corollary of Rolle's theorem simply states that for a real polynomial $f(x)$, roots of $f'(x)$ interlace those of $f(x)$.
It seems that by using similar line of proofs for the Rolle's theorem, one can get the following simple but interesting lemma (see Lemma 4.2 in [2]).

Lemma 1.1. *Let $\{f_i(x)\}_i$ be a finite collection of polynomials with all real coefficients, a positive leading coefficient and at least one real root. For each i, we also let R_i and r_i be the largest and second largest real roots (respectively) of $f_i(x)$. In the case $f_i(x)$ has only one real root, by convention we put $r_i = -\infty$. If $\max_j r_j \leq \min_j R_j$, then $f(x) = \sum_j f_j(x)$ has a real root R which satisfies $\min_j R_j \leq R$.*

The importance of the above *key lemma* is that it simply gives interesting necessary conditions for having a real root for *sums of polynomials*.
It is worth to note that the first derivative of some graph polynomials like matching and characteristic polynomials can be expressed as the sum of these polynomials restricted to their vertex-deleted subgraphs [3]. Hence, Lemma 1.1 can help us to analyze their property of having only real roots.
In this paper, we will concentrate on the clique polynomial [5] of a graph. By applying the key lemma to the combinatorial interpretations of the first and second derivatives of clique polynomials, we obtain several classes of graphs which have only clique roots. We finally conclude the paper with some open questions and conjectures.

[1]speaker

2 Main Results

Throughout this paper, we will assume that our graphs are all finite, simple and undirected. For the definitions which are not appear here, one may refer to [4].
A subset of vertices of a graph G that are pairwise adjacent is called a *complete* subgraph or a *clique* of G. A clique with k vertices is called a k-*clique*. The number of k-cliques will be denoted by $c_k(G)$. For a subset S of vertices, the graph with vertex set S and edges with end-vertices only on S is called an *induced subgraph* of G and denoted by $G[S]$. The set of vertices adjacent to a vertex v is called the (open) *neighborhood* of v and is denoted by $N(v)$. The subgraph obtained by deleting the vertex v from G will be denoted by $G - v$. In a similar way, a subgraph obtained by only removing an edge e from G is denoted by $G - e$.
A *chord* of a cycle is an edge connecting two *non-adjacent* vertices in the cycle. A *chordal* graph is a graph that any cycle of length greater than *three* has a chord.
For a given graph $G = (V, E)$, the *ordinary generating* function of the number of cliques of G is called the *clique polynomial* [5] of G and is denoted by $C(G, x)$. More precisely, we have

$$C(G, x) = \sum_{k=0}^{\omega(G)} c_k(G)x^k, \tag{1}$$

where $\omega(G)$ is the size of the largest clique of G. By convention, we may assume $c_0(G) = 1$ for any graph G. The real root of the clique polynomial of G is called the *clique root* of G.
The following recurrence relations can be obtained for clique polynomials using simple counting arguments [5]. Here, we use the notation $N(e) = N(u) \cap N(v)$ for the edge $e = \{u, v\}$.

$$C(G, x) = C(G - v, x) + xC(G[N(v)], x), \tag{2a}$$

$$C(G, x) = C(G - e, x) + x^2C(G[N(e)], x). \tag{2b}$$

The following combinatorial interpretations of the first and second derivatives of clique polynomial are given in [6].

$$\frac{d}{dx}C(G, x) = \sum_{v\in V(G)} C(G[N(v)], x), \tag{3a}$$

$$\frac{1}{2!}\frac{d^2}{dx^2}C(G, x) = \sum_{e\in V(G)} C(G[N(e)], x). \tag{3b}$$

It is also important to note that based on the recursive definition of chordal graphs using the idea of *pasting two complete graphs along a clique* and formula (2a), one can show that any k-connected chordal graphs has a clique root -1 of multiplicity k [7].
Before stating our first result, we need another easy but key statement.

Proposition 2.1. *Let T be a* tree *on $n > 1$ vertices. Then, the graph T has only clique roots. Moreover, the largest and the second largest clique roots are $R = -\frac{1}{n-1}$ and $r = -1$, respectively.*

An immediate corollary of Lemma 1.1 and Proposition 2.1 is the following.

Corollary 2.2. *Let F be any forest. Then F has only clique roots. Moreover $r_F \leq -1 \leq R_F$, where r_F and R_F denotes the smallest and largest clique root of F, respectively.*

Next, we present our main results of this paper which are the characterization of some classes of graphs with only clique roots.

Proposition 2.3. *Let G be a triangle-free graph. Then G has only clique roots.*

Proof. Since G is triangle-free, the neighborhood of each vertex is an independent set. That is $C(G[N(v)],x) = 1 + r_v x$ where r_v is the size of the corresponding independent set of v. If r denotes the smallest size of all those independent sets, then we clearly have

$$-\infty = \max_j r_j < -1 \leq -\frac{1}{r} = \min_j R_j. \tag{4}$$

Hence the conditions of Lemma 1.1 are true and the polynomial

$$\sum_{v \in V(G)} C(G[N(v)], x)$$

has a real root $-1 \leq R < 0$. Now considering (2a), we immediately conclude that $\frac{d}{dx}C(G,x)$ has a real root. Thus the graphs G has only clique roots. $\square$

Proposition 2.4. *Let G be a K_4-free connected chordal graph. Then G has only clique roots.*

Sketch of Proof. Since the neighborhood of each vertex v is a *forest* F_v, then considering Corollary 2.2 we conclude that $\max_{v \in V} r_{F_v} \leq \min_{v \in V} R_{F_v}$. Thus the key lemma implies that $\frac{d}{dx}C(G,x)$ has a real root. Now considering the fact $C(G,x)$ has a clique root -1 and $\frac{d}{dx}C(G,x)$ has a real root, the proof is complete. $\square$

Proposition 2.5. *Let G be a bi-connected K_5-free chordal graph. Then G has only clique roots.*

Sketch of Proof. considering the fact that $C(G,x)$ is a quartic polynomial with at least two clique roots $r = -1$, we just need to prove $\frac{d^2}{dx^2}C(G,x)$ has a real root. Now, since for each $e \in E(G)$ the graph $G[N(e)]$ is triangle-free, based on combinatorial formula (3b) and Lemma 1.1 the proof is complete. $\square$

3 Open Questions and Conjectures

Considering Lemma 1.1, we come up with the following conjecture.
Conjecture 1. Let G be a connected K_4-free graph, then G has only clique roots.
Based on Proposition 2.5, the following open question naturally arises.
Question 1. Is there any 2-connected non-chordal K_5-free graph which has only clique roots?
Consider the wheel graph W_5 on five vertices with an extra edge connecting two non-adjacent vertices on the outer ring. We will denote the resulting graph by W_5^+. Now it is clear that this graph is not chordal and $C(W_5^+, x) = (1+x)(1+5x+6x^2+x^3)$ has only real roots.
Conjecture 2. The class of all 2 - connected K_5-free graphs in which each edge belongs to at most two triangles has only clique roots.
Question 2. Which classes of K_r-free chordal graphs have only real roots.
We finally made the following stronger conjecture.
Conjecture 3. The class of l-connected chordal graphs which are K_{l+3}-free has only clique roots.
We believe that the last conjecture is a starting point to find another algebraic graph-theoretic proof of the well-known Turán's graph theorem.

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On some lower bounds for graph energy

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Abstract

The energy of a graph G, denoted by $\mathcal{E}(G)$, is defined as the sum of absolute values of all eigenvalues of G. There are many lower and upper bounds for the energy of graphs in terms of graph parameters. Here, we present some of them.

Keywords: Adjacency matrix, Eigenvalue of graphs, Graph energy.
Mathematics Subject Classification [2010]: 05C50, 15A18, 15A23

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. By *order* of G, we mean the number of vertices of G. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

Let G be a graph and $V(G) = \{v_1, \dots, v_n\}$. The *adjacency matrix* of G, $A(G) = [a_{ij}]$, is an $n \times n$ matrix, where $a_{ij} = 1$ if $v_iv_j \in E(G)$, and $a_{ij} = 0$, otherwise. Thus $A(G)$ is a symmetric matrix and all eigenvalues of $A(G)$ are real. By eigenvalues of a graph G, we mean the eigenvalues of $A(G)$. The *spectral radius* of G is defined as $s = \max\{|\lambda| : \lambda \in \mathrm{Spec}(G)\}$. The *energy* of a graph G, $\mathcal{E}(G)$, is defined as the sum of absolute values of eigenvalues of G. The concept of graph energy was first introduced by Gutman in 1978, see [6]. For more properties of the energy of graphs, the reader is referred to [7].

2 Main Results

Some lower bounds for the energy of graphs have been obtained by several authors. For quadrangle-free graphs, Zhou [10] studied the problem of bounding the graph energy in terms of the minimum degree together with other parameters. In [8], it was proved that for a connected graph G, $\mathcal{E}(G) \geq 2\delta(G)$ and the equality holds if and only if G is a complete multipartite graph with the equal size of parts. In [2], this lower bound was improved by showing that if G is a connected graph with average degree $\overline{d}$, then $\mathcal{E}(G) \geq 2\overline{d}$ and the equality holds if and only if G is a complete multipartite graph with the equal size of parts. Also in [2], the following conjecture proposed.

Conjecture.[2] *For every graph G whose adjacency matrix is non-singular, $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$ and the equality holds if and only if G is a complete graph.*

Here, we give some lower bounds for the energy of graphs.

Caporossi et al. gave the following lower bound based on the number of edges in [3] as

$$\mathcal{E}(G) \geq 2\sqrt{m}$$

[1]speaker

with equality if and only if G is the union of a complete bipartite graph $K_{a,b}$, where $ab = m$ and arbitrarily many isolated vertices.

The following lower bound in terms of the order and size of G and $\det A(G)$ was established in [9] as follows:

$$\mathcal{E}(G) \geq \sqrt{2m + n(n-1)|\det A(G)|^{\frac{2}{n}}}.$$

Also for bipartite graphs this lower bound was improved in [5] as

$$\mathcal{E}(G) \geq \sqrt{4m + n(n-2)|\det A(G)|^{\frac{2}{n}}}.$$

Although in [1], it has been shown that this lower bound holds for all connected graphs of order at least 7.

In [4], Das et al. derived the following lower bound for a graph involving the parameters order, size, spectral radius of G and $r = \min\{|\lambda| : \lambda \in \mathrm{Spec}(G)\}$ as

$$\mathcal{E}(G) \geq \frac{srn + 2m}{s + r},$$

with equality if and only if $|\lambda_i| = r$ or $|\lambda_i| = s$, for $i = 1, \ldots, n$.

Further in [1] the following lower bound obtained.

$$\mathcal{E}(G) \geq \frac{1}{2}\left(r(n-2) + \sqrt{r^2(n-2)^2 + 16m}\right).$$

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Perfect Roman {3}-domination of graphs

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Abstract

For a graph $G = (V, E)$ with $V = V(G)$ and $E = E(G)$, a perfect Roman {3}-dominating function is a function $f : V \to \{0, 1, 2, 3\}$ having the property that $3 \le \sum_{u \in N_G[v]} f(u) \le 4$, if $f(v) \in \{0, 1\}$ for any vertex $v \in G$. The weight of a perfect Roman {3}-dominating function f is the sum $f(V) = \sum_{v \in V(G)} f(v)$ and the minimum weight of a perfect Roman {3}-dominating function on G is the perfect Roman {3}-domination number of G, denoted by $\gamma_{\{PR3\}}(G)$. In this manuscript we study the perfect Roman {3}-domination of some graphs.

Keywords: Roman {3}-domination, Perfect Roman {3}-domination.

Mathematics Subject Classification [2010]: Primary: 05C69

1 Introduction

Let $G = (V, E)$ be a simple undirected graph of order n with $V = V(G)$ and $E = E(G)$. A set $S \subseteq V$ in a graph G is called a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set in G, and a dominating set of G of cardinality $\gamma(G)$ is called a γ-set of G. A subset $S \subseteq V$ is a k*-dominating set* if every vertex of $V - S$ has at least k neighbors in S. The k*-domination number* $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G (see [4]). Given a graph G and a positive integer $m \ge 2$, assume that $g : V(G) \to \{0, 1, 2, \ldots, m\}$ is a function, and suppose that $(V_0, V_1, V_2, \ldots, V_m)$ is the ordered partition of V induced by g, where $V_i = \{v \in V : g(v) = i\}$ for $i \in \{0, 1, \ldots, m\}$. So we can write $g = (V_0, V_1, V_2, \ldots, V_m)$. A *Roman dominating function* on G is a function $f : V \to \{0, 1, 2\}$ such that if $v \in V_0$ for some $v \in V$, then there exists a vertex $w \in N(v)$ with $w \in V_2$. The weight of a Roman dominating function is the sum $w_f = \sum_{v \in V(G)} f(v)$, and the minimum weight of w_f for every Roman dominating function f on G is called the *Roman domination number* of G, denoted by $\gamma_R(G)$.

Roman domination was introduced by Cockayne et al. in [2], although this notion was inspired by the work of ReVelle et al. in [7], and Stewart in [8]. The original study of Roman domination was motivated by the defense strategies used to defend the Roman Empire during the reign of Emperor Constantine the Great, 274-337 A.D. He decreed that for all cities in the Roman Empire, at most two legions should be stationed. Further, if a location having no legions was attacked, then it must be within the vicinity of at least one city at which two legions were stationed, so that one of the two legions could be sent to defend the attacked city. This part of history of the Roman Empire gave rise to the mathematical concept of Roman domination, as originally defined and discussed by Stewart [8] in (1999), and ReVelle and Rosing [7] in (2000).

[1]speaker

Beeler et al. [1] have recently defined double Roman domination. What they propose is a stronger version of Roman domination that doubles the protection by ensuring that any attack can be defended by at least two legions. In Roman domination at most two Roman legions are deployed at any one location. But as we will see in what follows, the ability to deploy three legions at a given location provides a level of defense that is both stronger and more flexible, at less than the anticipated additional cost.

A *double Roman dominating function* on a graph G is a function $f : V \to \{0, 1, 2, 3\}$ such that the following conditions are met:
(a) if $f(v) = 0$, then vertex v must have at least two neighbors in V_2 or one neighbor in V_3.
(b) if $f(v) = 1$, then vertex v must have at least one neighbor in $V_2 \bigcup V_3$.
The weight of a double Roman dominating function f on G is the sum $w_f = \sum_{v\in V(G)} f(v)$, and the minimum weight of w_f for every double Roman dominating function f on G is called *double Roman domination number* of G. We denote this number with $\gamma_{dR}(G)$ and a double Roman dominating function of G with weight $\gamma_{dR}(G)$ is called a $\gamma_{dR}(G)$-function of G. Although there exist research works on other parameters related to double Roman domination, such as total double Roman domination, independent double Roman domination, for example see [3, 5].

In [6] Mojdeh et al. define a variant of double Roman dominating functions, namely, Roman $\{3\}$-dominating functions that is an optimization of double Roman dominating function that has already been mentioned. These functions are defined as follows.
For a graph G, a Roman $\{3\}$-dominating function is a function $f : V \to \{0, 1, 2, 3\}$ having the property that for every vertex $u \in V$, if $f(u) \in \{0, 1\}$, then $f(N[u]) \geq 3$. Formally, a Roman $\{3\}$-dominating function $f : V \to \{0, 1, 2, 3\}$ has the property that for every vertex $v \in V$, with $f(v) = 0$, there exist at least either three vertices in $V_1 \cap N(v)$ or one vertex in $V_1 \cap N(v)$ and one in $V_2 \cap N(v)$ or two vertices in $V_2 \cap N(v)$ or one vertex in $V_3 \cap N(v)$ and for every vertex $v \in V$, with $f(v) = 1$, there exist at least either two vertices in $V_1 \cap N(v)$ or one vertex in $(V_2 \cup V_3) \cap N(v)$. The weight of a Roman $\{3\}$-dominating function is the sum $w_f = f(V) = \sum_{v\in V} f(v)$, and the minimum weight of a Roman $\{3\}$-dominating function f is the Roman $\{3\}$-domination number, denoted by $\gamma_{\{R3\}}(G)$.

A variant of Roman dominating functions, namely, perfect Roman $\{3\}$-dominating functions or perfect double Italian dominating function as follows. Hereafter we use the title Roman $\{3\}$ instead of double Italian.

Definition 1.1. A perfect Roman $\{3\}$-dominating function is a function $f : V \to \{0, 1, 2, 3\}$ having the property that $3 \leq \sum_{u\in N_G[v]} f(u) \leq 4$, if $f(v) \in \{0, 1\}$ for any vertex $v \in G$. The weight of a perfect Roman $\{3\}$-dominating function f is the sum $f(V) = \sum_{v\in V(G)} f(v)$ and the minimum weight of a perfect Roman $\{3\}$-dominating function on G is the perfect Roman $\{3\}$-domination number of G, denoted by $\gamma_{\{PR3\}}(G)$.

Here we introduce some notations and terminologies.
The *open neighborhood* of a vertex $v \in V(G)$ is the set $N(v) = \{u : uv \in E(G)\}$. The *closed neighborhood* of a vertex $v \in V(G)$ is $N[v] = N(v)\cup\{v\}$. The open neighborhood of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S}N(v)$. The closed neighborhood of a set $S \subseteq V$ is the set $N[S] = N(S) \cup S = \cup_{v\in S}N[v]$. We denote the *degree* of v by $d_G(v) = |N(v)|$. Given a set $S \subseteq V$, the *private neighborhood* $pn[v, S]$ of $v \in S$ is defined by $pn[v, S] = N[v] - N[S - \{v\}]$, equivalently, $pn[v, S] = \{u \in V : N[u] \cap S = \{v\}\}$. Each vertex in $pn[v, S]$ is called a private neighbor of v. By $\Delta = \Delta(G)$ and $\delta = \delta(G)$, we denote the maximum degree and minimum degree of a graph G, respectively. We write K_n, P_n and C_n for the complete graph, path and cycle of order n, respectively. A tree T is an acyclic connected graph (see [9]).

2 Results

In this section we study the perfect Roman $\{3\}$ domination of certain graphs.

Proposition 2.1. *Let* $n \geq 1$. *Then* $\gamma_{\{PR3\}}(P_n) = \begin{cases} n & \text{if } n \equiv 0 \pmod 3 \\ n+1 & \text{otherwise} \end{cases}$.

Proposition 2.1 shows that for any path P_n, $\gamma_{\{PR3\}}(P_n) = \gamma_{R3}(P_n)$.

Proposition 2.2. *For a cycle C_n, we have $\gamma_{\{PR3\}}(C_n) = n$*

We have the following results about Perfect Roman {3}-domination of complete r-partite graphs with $r \geq 2$.

Proposition 2.3. *For any complete bipartite graph we have.*
1. $\gamma_{\{PR3\}}(K_{1,n}) = \gamma_{\{R3\}}(K_{1,n}) = 3$,
2. $\gamma_{\{PR3\}}(K_{2,n}) = \gamma_{\{R3\}}(K_{2,n}) = 4$ *for* $n \geq 2$,
3. $\gamma_{\{PR3\}}(K_{3,n}) = \gamma_{\{R3\}}(K_{3,n}) = 6$, *for* $n \geq 3$,
4. $\gamma_{\{R3\}}(K_{m,n}) = \gamma_{\{R3\}}(K_{m,n}) = 6$ *for* $m, n \geq 4$.

Proposition 2.4. *Let $G = K_{n_1,n_2,\ldots,n_r}$ be the complete r-partite graph with $r \geq 3$ and $n_1 \leq n_2 \leq \ldots \leq n_r$.*
1. *If $n_1 = 1$, then $\gamma_{\{PR3\}}(G) = 3$,*
2. *If $n_1 = 2$, then $\gamma_{\{PR3\}}(G) = 4$,*
3. *If $n_1 \geq 3$ and $r \geq 4$, then $\gamma_{\{PR3\}}(G) = 4$,*
4. *If $n_1 \geq 3$ and $r = 3$, then $\gamma_{\{PR3\}}(G) = 5$.*

3 Small perfect Roman {3}-domination

A simple characterization of graphs whose perfect Roman {3}-domination is at lest 3. The following is straight forward.

Proposition 3.1. *If G is a graph of order $n \geq 2$, then $\gamma_{\{PR3\}}(G) \geq 3$, with equality if and only if $\Delta(G) = n - 1$.*

Next we investigate the relation between small perfect Roman {3}-domination with maximum degree, minimum degree and order of graphs.

Theorem 3.2. *If G is a graph with $\delta(G) = \delta \geq 2$ and $\Delta(G) = \Delta \leq 3$, then $\gamma_{PR3}(G) \leq |V(G)| + 2 - \delta$, and this bound is sharp.*

As an immediate consequence, one can see that if G is a cubic graph, then $\gamma_{\{PR3\}}(G) \leq n - 1$.
The following result is proved with the same method of the proof of Proposition 10 of [6].

Proposition 3.3. *Let G be a connected graph of order $n \geq 3$. Then $\gamma_{\{PR3\}}(G) = 4$ if and only if $\Delta(G) \leq n - 2$ and*
- *$G = \overline{K_2} \vee H$, where H is a graph with $\Delta(H) \leq |V(H)| - 2$, or*
- *there is a cycle C_4 with vertices $\{v_1, v_2, v_3, v_4\}$ such that $G = H \cup C_4$ and every vertex of H is adjacent to at least 3 vertices of C_4 and G does not have two non adjacent vertices u, w such that $\deg(u) = \deg(w) = n - 2$ or*
- *there is graph $K_4 - e$ with vertices $\{v_1, v_2, v_3, v_4\}$ such that $G = H \cup (K_4 - e)$ and every vertex of H is adjacent to at least 3 vertices of $K_4 - e$ and if $e = v_1v_3$, then $\deg_G(v_2)$ and $\deg_G(v_4)$ must be at least 4, or*
- *there is complete graph K_4 with vertices $\{v_1, v_2, v_3, v_4\}$ such that $G = H \cup K_4$ and every vertex of H is adjacent to at least 3 vertices of K_4 and $\deg(v_i) \geq 4$, $(1 \leq i \leq 4)$.*

one of the main results is as follows.

Proposition 3.4. *Let G be a connected graph of order $n \geq 3$. Then $\gamma_{\{PR3\}}(G) = 5$ if and only if one of the followinhg holds:*
1. *There is a path $P_3 = u_1u_2u_3$ such that $G = P \cup H$, where $P \in \{P_3, P_3 - e\}$ and every vertex of H is adjacent to exactly 2 vertices of P and if $P = P_3$, we must have $d(u_2) \leq n - 2$.*
2. *$G = C_3 \cup H$ such that there exists $a \in V(H)$ such that for every $x \in N_H(a)$, x is adjacent to 1 or 2 vertices of C_3 and a is not adjacent to any vertex of C_3 and for every $x \notin N_H(a)$, x is adjacent to all vertices*

of C_3.
3. $G = C_4 \cup H$, *where* $\{2\} \subseteq \{|N_{C_4}(x)| \mid x \in H\} \subseteq \{2,3\}$ *and* $\bigcap_{x \in A_2} N_{C_4}(x) \neq \emptyset$.
4. $G = C \cup H$, *where* $C \in \{C_5, C_5 + e\}$ *such that every vertex of* H *is adjacent to* 3 *or* 4 *vertices in* C.
5. $G = (C_5 + \{e,f\}) \cup H$, *where* $\Delta(C_5 + \{e,f\}) = 3$ *, each vertex of* H *is adjacent to exactly* 3 *or* 4 *vertices in* $C_5 + \{e,f\}$ *and if* $e = v_1v_3$ *and* $f = v_2v_5$, $d(v_1), d(v_2) \geq 4$.

Proof. Let $f = (V_0, V_1, V_2, V_3)$ be a γ_{PR3}−function of G. By definition, $|V_1| + 2|V_2| + 3|V_3| = 5$. Hence $(|V_1|, |V_2|, |V_3|) \in \{(1,2,0), (0,1,1), (3,1,0), (5,0,0)\}$.
1. In this case, $(|V_1|, |V_2|, |V_3|) = (1,2,0)$. So, there exist one vertex u_1 with assigned 1 and two vertices u_2, u_3 with assigned 2.
2. In this case, $(|V_1|, |V_2|, |V_3|) = (3,1,0)$. So, there exist three vertices v_1, v_2, v_3 with assigned 1 and one vertex v_4 with assigned 2. Let the induced subgraph by v_is be $C_3 \cup K_1$. It suffices put $\{a\} = V(K_1)$.
3. In this case, $(|V_1|, |V_2|, |V_3|) = (3,1,0)$ and the induced subgraph by v_is(as in part 2) is C_4.
for the parts 4, 5, suppose $(|V_1|, |V_2|, |V_3|) = (5,0,0)$. □

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The Tutte polynomial of matroids constructed by a family of splitting operations

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Abstract

To extract some more information from the constructions of matroids that arise from new operations, computing the Tutte polynomial, plays an important role. In this paper, we consider applying three operations of splitting, element splitting and splitting-off to a binary matroid and then introduce the Tutte polynomial of resulting matroids by these operations in terms of that of original matroids.

Keywords: Tutte polynomial, Splitting, Splitting off, Element splitting.
Mathematics Subject Classification [2010]: Primary: 05B35

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1 Introduction

The matroid notations and terminology used here will follow [5]. Raghunathan, Shikare, and Waphare [6] extended the splitting operation and Azadi ([9],[1]) extended the splitting off and the element splitting operations from graphs to binary matroids. These operations are defined as follows.

Definition 1.1. Let M be a binary matroid on a set E and A be a matrix that represents M over $GF(2)$. Consider two elements x and y of $E(M)$. Let $A_{x,y}$ be the matrix that is obtained by adjoining an extra row to A whose entries are zero everywhere except in the columns corresponding to x and y. Let $A'_{x,y}$ be the matrix that is obtained by adjoining an extra column to $A_{x,y}$ with this column being zero everywhere except in the last row. Finally, let A_{xy} be the matrix that is obtained by adjoining an extra column to A which is the sum of the columns corresponding to x and y, and then deleting the two columns corresponding to x and y. Let $M_{x,y}$, $M'_{x,y}$ and M_{xy} be the matroids that represented by the matrices $A_{x,y}$, $A'_{x,y}$ and A_{xy}, respectively. Then the transition from M to $M_{x,y}$, $M'_{x,y}$ and M_{xy} is called *the splitting operation*, *element splitting operation* and *splitting off operation* (or in short *split-off*), respectively.

Definition 1.2. [10] Let A be a matrix that represents a binary matroid M. A non-loop element x of M is said to be *equivalent* to another non-loop element y of M and denoted by $x \sim y$, if there is a row in the row space of A with entry 1 in the columns corresponding to x and y, and entry 0 elsewhere.

Throughout this paper, for notational convenience, letting α and β be the labels of new elements that are added to an original matroid after applying split-off and element splitting operations, respectively. We will use the following propositions to prove our main results in the next section. One can find the gathered parts of these propositions in [10],[1] and [6]

[1]speaker

Proposition 1.3. *Let M be a binary matroid and let $x, y \in E(M)$. Then*

i) $M'_{x,y}/\beta = M$;

ii) $M'_{x,y} \setminus \beta = M_{x,y}$ *and* $M_{x,y}/x \cong M_{x,y}/y \cong M_{xy}$;

iii) $M'_{x,y} \setminus \{\beta, x, y\} = M_{x,y} \setminus x/y = M_{x,y} \setminus y/x = M_{x,y} \setminus \{x, y\} = M_{xy} \setminus \alpha = M \setminus \{x, y\}$. *Moreover, the sets* $\{x, y\}$ *and* $\{\beta, x, y\}$ *are cocircuits of* $M_{x,y}$ *and* $M'_{x,y}$, *respectively;*

iv) y *is a coloop of* $M_{x,y} \setminus x$ *while* x *is a coloop of* $M_{x,y} \setminus y$;

v) *If* $x \sim y$ *in* M, *then* $M_{xy} \cong M/\{x\} \cong M/\{y\}$ *and* $M_{xy}/\{\alpha\} = M/\{x, y\}$;

vi) *If at least one of* x *and* y *is a coloop of* M, *then* α *is a coloop of* M_{xy} *and both of* x *and* y *are coloop of* $M_{x,y}$.

Let M be a matroid and X be a subset of $E(M)$. We denote by $M(X)$, The restriction of M with respect to X which is the matroid $M \setminus (E - X)$.

Proposition 1.4. *Let r and r' be the rank functions of the matroids M and M_{xy}, respectively. Let $X \subseteq E(M_{xy})$. Then*

i) *If* $\alpha \notin X$, *then* $r'(X) = r(X)$;

ii) *if* $\alpha \in X$ *and* α *is not a coloop of* $M_{xy}(X)$, *then* $r'(X) = r(X - \alpha)$;

iii) *if* $\alpha \in X$ *and* α *is a coloop of* $M_{xy}(X)$, *then* $r'(X) = r(X - \alpha) + 1$.

The Tutte polynomial is a two-variable polynomial originally defined for graphs by Tutte and Whitney and later generalized to matroids by Crapo [3]. It was first conceived as an extension of the chromatic polynomial, but nowadays it is known to have applications in many areas of combinatorics and other areas of mathematics.

Definition 1.5. The Tutte polynomial $T(M, u, v)$ (or briefly $T(M)$) of a matroid M on the set E is given by

$$T(M) = \sum_{A \subseteq E} (u - 1)^{z(A)} (v - 1)^{n(A)}.$$

Where $z(A) = r(M) - r(A)$ and $n(A) = |A| - r(A)$.

Computing the Tutte polynomial of a matroid is known to be #P-hard, so some formulas are presented to reduce this computation to simpler computations. The next proposition shows some of the well-known formulas for the Tutte polynomial of a given matroid.

Proposition 1.6. [2] *Let M be a matroid. Then the Tutte polynomial $T(M)$ has the following properties*

i) $T(M, u, v) = T(M^*, v, u)$.

ii) *Given a matroid* N *with* $M \cong N$, *we have* $T(M) = T(N)$.

iii) *If* e *is an element of* M *and* e *is neither a loop nor a coloop of* M, *then*

$$T(M) = T(M \setminus e) + T(M/e).$$

iv) *If* e *is a loop of* M, *then* $T(M) = v\big(T(M \setminus e)\big)$.

v) *If* e *is a coloop of* M, *then* $T(M) = u\big(T(M \setminus e)\big)$.

vi) *If* $M = M_1 \oplus M_2$, *then* $T(M) = T(M_1)T(M_2)$.

2 Main Results

In [8], the authors find the following relation between a given binary matroid M and $M_{x,y}$.

$$T(M_{x,y}, 2, 1) \equiv T(M, 2, 1) \pmod 2.$$

Where $T(M, 2, 1)$ and $T(M_{x,y}, 2, 1)$ count the number of independent sets in M and $M_{x,y}$,respectively. Now we want to find some splitting formulas to compute the Tutte polynomial of a matroid that is obtained by splitting, element splitting and split-off operation in terms of the Tutte polynomial of original matroid or in terms of that of each other.

Theorem 2.1. *Let M be a binary matroid and Let $x, y \in E(M)$ such that $x \nsim y$. Then*

$$T(M'_{x,y}) = (u+1)T(M \setminus \{x, y\}) + T(M_{xy}/\alpha) + T(M).$$

Corollary 2.2. *Let M be a binary matroid and Let $x, y \in E(M)$. Then*

$$T(M_{x,y}) = (u+1)T(M \setminus \{x, y\}) + T(M_{xy}/\alpha).$$

Theorem 2.3. *The Tutte polynomial of a split-off matroid M_{xy} from given binary matroid M with $x, y \in E(M)$ has the following properties:*

i) *If at least one of x and y is a coloop of M, then $T(M_{xy}) = uT(M \setminus \{x, y\})$.*

ii) *If $\{x, y\}$ is a circuit of M, then $T(M_{xy}) = vT(M \setminus \{x, y\})$.*

iii) *If $\{x, y\}$ is a cocircuit of M which is not a circuit of M, then*

$$T(M_{xy}) = T(M) - uT(M \setminus \{x, y\}).$$

Corollary 2.4. *Let M be a binary matroid and Let $x, y \in E(M)$. Then*

i) *If exactly one of x and y is a coloop of M, then*
$T(M_{x,y}) = u^2T(M \setminus \{x, y\})$ and $T(M'_{x,y}) = u^2T(M \setminus \{x, y\}) + T(M)$.

ii) *If $\{x, y\}$ is a circuit of M, then*
$T(M_{x,y}) = (u+v)T(M \setminus \{x, y\})$ and $T(M'_{x,y}) = (u+v)T(M \setminus \{x, y\}) + T(M)$.

iii) *If $x \sim y$, then $T(M_{x,y}) = T(M)$ and $T(M'_{x,y}) = uT(M)$*

Given a binary matroid M, the Theorem 2.3 introduces Tutte polynomial of M_{xy} when $x \sim y$ or $\{x, y\}$ is a circuit of M or exactly one of x and y is a coloop of M. Now we want to provide a splitting formula for such a Tutte polynomial when $x \nsim y$.

Let M_{xy} be the split-off of a binary matroid M with $x, y \in E(M)$ and $x \nsim y$ in M. Then there is a partition of the collection of all subsets of $E(M_{xy})$ into non-empty subsets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ such that
-$\mathcal{A}_1 = \{A \subset E(M_{xy}) : \alpha \notin A\}$;
-$\mathcal{A}_2 = \{A \subseteq E(M_{xy}) : \alpha$ is not a coloop of $M_{xy}(A)\}$;
-$\mathcal{A}_3 = \{A \subseteq E(M_{xy}) : \alpha$ is a coloop of $M_{xy}(A)\}$. Clearly, any of $\mathcal{A}_1$ and $\mathcal{B}_1 \cup \mathcal{B}_2$ where

$$\mathcal{B}_i = \{Z - \alpha : Z \in \mathcal{A}_i, \text{ for } i \in \{2, 3\}\}$$

are collection of all subsets of $E(M \setminus \{x, y\})$.
Therefore

$$T(M \setminus \{x, y\}) = \sum_{i=1}^{2} \sum_{A \in \mathcal{B}_i} (u-1)^{z(A)}(v-1)^{n(A)}.$$

We denote two non-zero terms of this summation formula by $T_i(M)$. Indeed,

$$T(M \setminus \{x, y\}) = T_1(M) + T_2(M).$$

The following theorem will be helpful in computing the Tutte polynomial of a matroid obtained by applying split-off operation on two non-equivalent elements of a given binary matroid. We use the notation above to prove it.

Theorem 2.5. *Let M be a binary matroid and $x, y \in E(M)$ such that $x \nsim y$ in M. Then the Tutte polynomial of M_{xy} can be computed by the following formula.*

$$T(M_{xy}) = vT_1(M) + \frac{u}{(u-1)}T_2(M).$$

The following lemmas are an immediate consequence of the last theorem.

Corollary 2.6. *Let M be a binary matroid and $x, y \in E(M)$ such that $x \nsim y$ in M. Then the Tutte polynomial of $M_{x,y}$ can be computed by the following formula.*

$$T(M_{x,y}) = (u+v)T_1(M) + \frac{u^2}{(u-1)}T_2(M).$$

Corollary 2.7. *Let M be a binary matroid and $x, y \in E(M)$ such that $x \nsim y$ in M. Then the Tutte polynomial of $M'_{x,y}$ can be computed by the following formula.*

$$T(M'_{x,y}) = (u+v)T_1(M) + \frac{u^2}{(u-1)}T_2(M) + T(M).$$

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Connected Graphs with Nullity Two are Non-hypoenergetic

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Abstract

The energy of a graph G, denoted by $\mathcal{E}(G)$, is defined as the sum of absolute values of all eigenvalues of G. A graph of order n, whose energy is less than n, i.e., $\mathcal{E}(G) < n$, is said to be hypoenergetic. Graphs for which $\mathcal{E}(G) \geq n$ are called non-hypoenergetic. A graph of order n is said to be orderenergetic, if its energy and its order are equal, i.e., $\mathcal{E}(G) = n$. It is proved that every connected graph with nullity 2 is non-hypoenergetic. In particular, there are only two connected orderenergetic graphs with nullity 2.

Keywords: Energy of a graph, Non-hypoenergetic graphs, Orderenergetic graphs, Nullity of a graph.

Mathematics Subject Classification [2010]: 05C50

1 Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. By *order* and *size* of G, we mean the number of vertices and the number of edges of G, respectively. The path and the cycle of order n are denoted by P_n and C_n, respectively. A complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. If $m = n$, we say $K_{m,m}$ is balanced. A $\{1,2\}$-*subgraph* of G is a subgraph which is a disjoint union of a matching and a 2-regular subgraph of G. A $\{1,2\}$-*subgraph* which is a spanning subgraph, is called a $\{1,2\}$-*factor*.

The *adjacency matrix* of G, $A(G) = [a_{ij}]$, is an $n \times n$ matrix, where $a_{ij} = 1$ if $v_iv_j \in E(G)$, and $a_{ij} = 0$, otherwise. Thus $A(G)$ is a symmetric matrix and all eigenvalues of $A(G)$ are real. Let $\eta(G)$, the *nullity* of G, denote the the number of zero eigenvalues of $A(G)$. The energy of a graph G, $\mathcal{E}(G)$, is defined as the sum of absolute values of eigenvalues of $A(G)$, see [5].

Graphs of order n, satisfying the condition $\mathcal{E}(G) < n$ are named *hypoenergetic* [8] and their properties were studied in [6, 7, 8]. Graphs for which $\mathcal{E}(G) \geq n$ are said to be *non-hypoenergetic*. A graph is called *orderenergetic*, if its energy and its order are equal, i.e., $\mathcal{E}(G) = n$. Some basic properties of orderenergetic graphs were studied in [2]. The authors showed that there are infinitely many connected orderenergetic graphs. They proved that a graph having a $\{1,2\}$-factor, is orderenergetic if and only if it is a disjoint union of balanced complete bipartite graphs. Also it was established that there is no orderenergetic graph with nullity 1. In this paper, we investigate some graphs whose energy exceeds the number of vertices. We prove that if a graph G, has a $\{1,2\}$-subgraph of order $n-1$, then $\mathcal{E}(G) > n$, except for $K_{r,r+1}$, $r \geq 1$. It is shown that if $K_{2,4}$ is an induced subgraph of a graph G such that $G \setminus V(K_{2,4})$ has a perfect matching and G has no component isomorphic to $K_{r,r+2}$, then $\mathcal{E}(G) > n$. It is also proved that if P_3 is an induced subgraph of a graph G, such that $G \setminus V(P_3)$ has a perfect matching and G has no component isomorphic to $K_{r,r+1}$, then $\mathcal{E}(G) > n$. We apply these facts, to classify connected orderenergetic graphs with nullity 2. We show that every graph with nullity 2 is non-hypoenergetic. In particular, except two graphs, the energy of each graph with nullity 2, exceeds the number of its vertices. The following lemmas are needed in the sequel.

[1]speaker

Lemma 1.1. *[3] Let G be a graph and $H_1, \ldots, H_k$ be its k vertex-disjoint induced subgraphs. Then*

$$\mathcal{E}(G) \geq \sum_{i=1}^{k} \mathcal{E}(H_i).$$

Lemma 1.2. *[9] Let H be an induced subgraph of a graph G. Then $\mathcal{E}(H) \leq \mathcal{E}(G)$ and equality holds if and only if $E(G) = E(H)$,*

Lemma 1.3. *[2] Let G be a graph of order n. If G has a $\{1,2\}$-factor. Then $\mathcal{E}(G) \geq n$. Equality holds if and only if G is the disjoint union of balanced complete bipartite graphs.*

Lemma 1.4. *[1, 2] If n is an odd integer, then $\mathcal{E}(C_n) \geq n + 1$. In particular, for $n \geq 9$, $\mathcal{E}(C_n) \geq n + 2$.*

Lemma 1.5. *[4] There are no connected orderenergetic graphs of odd order.*

Lemma 1.6. *[2] There is no connected orderenergetic graph with $\eta = 1$.*

2 Main Results

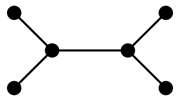
In this section, we investigate some graphs whose energy exceeds the number of vertices. We show that every graph with nulity 2 is non-hypoenergetic. In particular, there are exactly two orderenergetic graphs with nullity 2.

Lemma 2.1. *Let G be a graph of order n such that $K_{2,4}$ is an induced subgraph of G and $G \setminus V(K_{2,4})$ has a perfect matching. If G has no component isomorphic to $K_{r,r+2}$ (for each r), then $\mathcal{E}(G) > n$.*

Lemma 2.2. *Let G be a graph of order n such that P_3 is an induced subgraph of G and $G \setminus V(P_3)$ has a perfect matching. If G has no component isomorphic to $K_{r,r+1}$ (for each r), then $\mathcal{E}(G) > n$.*

Corollary 2.3. *Let G be a connected graph of order n. If G has a $\{1,2\}$-subgraph of order $n-1$, then $\mathcal{E}(G) > n$, except for $G = K_{r,r+1}$, $r \geq 1$. In particular, if the nullity of G is 1, then $\mathcal{E}(G) > n$, except for $K_{1,2}$.*

Theorem 2.4. *Let G be a connected graph of order n with nullity 2. If G is not C_4 or the follwong graph, then $\mathcal{E}(G) > n$. In particular, C_4 and the following graph are the only orderenergetic graphs with nullity 2.*



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The spectrum of a class of graphs derived from Grassmann graphs

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Abstract

Let n, k be positive integers such that $n \geq 3$, $k < \frac{n}{2}$. Let q be a power of a prime p and $\mathbb{F}_q$ be a finite field of order q. Let $V(q, n)$ be a vector space of dimension n over $\mathbb{F}_q$. We define the graph $S(q, n, k)$ as a graph with the vertex set $V = V_k \cup V_{k+1}$, where V_k and V_{k+1} are subspaces in $V(q, n)$ of dimension k and $k+1$ respectively, in which two vertices v and w are adjacent whenever v is a subspace of w or w is a subspace of v. It is clear that the graph $S(q, n, k)$ is a bipartite graph. In this paper, we study some properties of this graph. In particular, we determine the spectrum of the graph $S(q, n, k)$.

Keywords: connected graph, Grassmann graph, spectrum
Mathematics Subject Classification [2010]: 05C50

1 Introduction

In this paper, a graph $\Gamma = (V, E)$ is considered as an undirected simple finite graph, where $V = V(\Gamma)$ is its vertex set and $E = E(\Gamma)$ is its edge set. For the terminology and notation not defined here, we follow [1, 2]. Let p be a positive prime integer and $q = p^m$ where m is a positive integer. Let n, k be positive integers with $k < n$. Let $V(q, n)$ be a vector space of dimension n over the finite field $\mathbb{F}_q$. Let V_k be the family of all subspaces of $V(q, n)$ of dimension k. Every element of V_k is also called a k-subspace. The Grassmann graph $G(q, n, k)$ is the graph with the vertex set V_k, in which two vertices u and w are adjacent if and only if $\dim(u \cap w) = k - 1$. Note that if $k = 1$, we have a complete graph, so we shall assume that $k > 1$. It is clear that the number of vertices of the Grassmann graph $G(q, n, k)$, that is $|V_k|$, is the Gaussian binomial coefficient,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})} = \frac{(q^n - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1) \cdots (q - 1)}.$$

Noting that $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$, it follows that $|V_k| = |V_{n-k}|$. It is easy to show that if $1 \leq i < j \leq \frac{n}{2}$, then $|V_i| < |V_j|$. It can be shown that $G(q, n, k) \cong G(q, n, n-k)$ [1], and hence in the sequel we assume that $k \leq \frac{n}{2}$. It is easy to see that the distance between two vertices v and w in this graph is $k - dim(v \cap w)$. The Grassmann graph is a distance-regular graph of diameter k [1]. Concerning the matter we have the following fact [1].

Theorem 1.1. *Let $V = V(q, n)$ be as above. Suppose $0 \leq i, j \leq n$. Then*
(i) The number of k-spaces of V is $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}$
(ii) If X is a j-space of V, then there are precisely $q^{ij} \begin{bmatrix} n-j \\ i \end{bmatrix}$ i-spaces Y in V such that $X \cap Y = 0$.
(iii) If X is a j-space of V, then there are precisely $q^{(i-m)(j-m)} \begin{bmatrix} n-j \\ i-m \end{bmatrix} \begin{bmatrix} j \\ m \end{bmatrix}$ i-spaces Y in V such that $X \cap Y$ is an m-space.

[1] speaker

Now, we can deduce the following fact from (iii) of Theorem 1.1

Corollary 1.2. *If X is a k-space of V, then there are precisely* $q^{(k+1-k)(k-k)}\begin{bmatrix}n-k\\k+1-k\end{bmatrix}\begin{bmatrix}k\\k\end{bmatrix}=\begin{bmatrix}n-k\\1\end{bmatrix}$, *$(k+1)$-spaces Y in V such that $X \leq Y$.*

Proof. We know that a $(k+1)$-space Y contains the k-space X if and only if $Y \cap X$ is the k-space X. Now the result follows from (iii) of Theorem 1.1 □

Definition 1.3. Let n, k be positive integers such that $n \geq 3$, $k < n$. Let q be a power of a prime p and $\mathbb{F}_q$ be a finite field of order q. Let $V(q,n)$ be a vector space of dimension n over $\mathbb{F}_q$. We define the graph $S(q,n,k)$ as a graph with the vertex-set $V = V_k \cup V_{k+1}$, where V_k and V_{k+1} are the family of subspaces in $V(q,n)$ of dimension k and $k+1$ respectively, in which two vertices v and w are adjacent whenever v is a subspace of w or w is a subspace of v.

It is clear that this graph is a bipartite graph with partition $V = V_k \cup V_{k+1}$,

$$V_k = \{v \subset V(q,n) | dim(v) = k\},$$

$$V_{k+1} = \{v \subset V(q,n) | dim(v) = k+1\}.$$

When $n = 2k+1$, the graph $S(q,n,k)$ is called a doubled Grassman graph [1] and some of its properties have been studied by some authors [3]. The automorphism group of the graph $S(q,n,k)$ has recently been determined [4]. We know that the number of k-subspaces of a vector space $V(q,n)$ is the Gaussian binomial coefficient $\begin{bmatrix}n\\k\end{bmatrix}_q = \begin{bmatrix}n\\k\end{bmatrix}$. Thus, $|V_k| = \begin{bmatrix}n\\k\end{bmatrix}$ and $|V_{k+1}| = \begin{bmatrix}n\\k+1\end{bmatrix}$, hence the order of the graph $S(q,n,k)$ is $\begin{bmatrix}n\\k\end{bmatrix} + \begin{bmatrix}n\\k+1\end{bmatrix}$. If $v \in V_k$, then from Corollary 1.2, it follows that $deg(v) = \begin{bmatrix}n-k\\1\end{bmatrix} = q^{n-k-1} + q^{n-k-2} + \cdots + q + 1$. We know that the number of k-subspaces of a vector space of dimension $k+1$ is the Gussion binomial coefficient $\begin{bmatrix}k+1\\k\end{bmatrix}$, hence if $v \in V_{k+1}$, then $deg(v) = \begin{bmatrix}k+1\\k\end{bmatrix}$. When $G = V_1 \cup V_2$, $G = V_1 \cap V_2 = \emptyset$ is a bipartite graph such that $|V_1| = n_1, |V_2| = n_2$, also each vertex in V_1 is of order r_1 and each vertex in V_2 is of order r_2, then the graph G is called a bi-regular bipartite graph with parameters (n_1, n_2, r_1, r_2). Thus the graph $S(q,n,k)$ is a bi-regular bipartite graph with parameters $(\begin{bmatrix}n\\k\end{bmatrix}, \begin{bmatrix}n\\k+1\end{bmatrix}, \begin{bmatrix}n-k\\1\end{bmatrix}, \begin{bmatrix}k+1\\k\end{bmatrix})$. It can be shown that this graph is a connected graph (Proposition 2.1). We can easily see that this graph is a regular graph, when $n = 3, k = 1$, since $r_1 = r_2 = q+1$ and in this case $|V_1| = |V_2| = q^2 + q + 1$. Noting that $\begin{bmatrix}n\\k\end{bmatrix} = \begin{bmatrix}n\\n-k\end{bmatrix}$, it is easy to show that $S(q,n,k) \cong S(q,n,n-k-1)$, hence in the sequel we assume $k < \frac{n}{2}$.

Let Γ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(\Gamma)$. The adjacency matrix $A = A(\Gamma) = [a_{ij}]$ of Γ is an $n \times n$ symmetric matrix of 0's and 1's with $a_{ij} = 1$ if and only if v_i and v_j are adjacent. The characteristic polynomial of Γ is the polynomial $P(G) = P(G,x) = det(xI_n - A)$, where I_n denotes the $n \times n$ identity matrix. The spectrum of $A(\Gamma)$ is also called the spectrum of Γ. If the eigenvalue of Γ are ordered by $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, and their multiplicities are m_1,m_2,…,m_r, respectively, then we write ;

$$Spec(\Gamma) = \begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_r\\ m_1, m_2, \ldots, m_r\end{pmatrix} \quad or \quad Spec(\Gamma) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_r^{m_r}\}$$

Let $\mathbb{F}_q$ be a finite field of order q and let $V(q,n)$ be an n-dimensional vector space over a field $\mathbb{F}_q$. The Grassmann graph $G(q,n,k)$ is a graph whose vertex-set is the family of k-subspaces of $V(q,n)$, in which two vertices v and w are adjacent if and only if $dim(v \cap w) = k-1$. Concerning the spectrum of the Grassmann graph $G(q,n,k)$, we have the following fact [1].

Theorem 1.4. *Let Γ be the Grassmann graph $G(n,q,k)$. Then Γ has diameter $d = min(k, n-k)$. Moreover, Γ has eigenvalues and multiplicities given by*

$$\theta_j = q^{j+1}\begin{bmatrix}k-j\\1\end{bmatrix}\begin{bmatrix}n-k-j\\1\end{bmatrix} - \begin{bmatrix}j\\1\end{bmatrix} \; and \; f_j = \begin{bmatrix}n\\j\end{bmatrix} - \begin{bmatrix}n\\j-1\end{bmatrix}$$

where $0 \leq j \leq d$.

2 Main Results

In this paper, we wish to determine the spectrum of the graph $Sp(q, n, k)$.

Proposition 2.1. *The graph $\Gamma = S(q, n, k)$ is a connected graph.*

We now proceed to determine the spectrum of the graph $Sp(q, n, k)$.

Theorem 2.2. *Let $V(q, n)$ be a vector space of dimension n over a field $\mathbb{F}_q$, where q is a power of a prime p. Let $k < \frac{n}{2}$ and $\Gamma = S(q, n, k)$. Then, the graph Γ has distinct eigenvalues $\pm\sqrt{\theta_j}$ and $\pm\sqrt{\gamma_i}$, where*

$$\theta_j = q^{j+1}\begin{bmatrix} k-j \\ 1 \end{bmatrix}\begin{bmatrix} n-k-j \\ 1 \end{bmatrix} - \begin{bmatrix} j \\ 1 \end{bmatrix} + \begin{bmatrix} n-k \\ 1 \end{bmatrix},$$

and

$$\gamma_i = q^{i+1}\begin{bmatrix} k+1-i \\ 1 \end{bmatrix}\begin{bmatrix} n-k-1-i \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} + \begin{bmatrix} k+1 \\ k \end{bmatrix}.$$

Moreover, the multiplicity of θ_j is $f_j = \begin{bmatrix} n \\ j \end{bmatrix} - \begin{bmatrix} n \\ j-1 \end{bmatrix}$, $0 \leq j \leq k$, and the multiplicity of γ_i is $e_i = \begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n \\ i-1 \end{bmatrix}$, $0 \leq i \leq k+1$.

Proof. Let A be the adjacency matrix of the graph Γ. In the first step, we determine the spectrum of the matrix A^2. Let $(A^2)_{v,w}$ be the entry in the vth row and wth column of A^2. Not that $(A^2)_{v,w}$ is the number of 2-paths between the vertices v and w. We have the following cases.
(i) If $v = w$, then $(A^2)_{v,w}$ is the number of neighbors of v, hence we have;

$$(A^2)_{v,v} = \begin{cases} \begin{bmatrix} n-k \\ 1 \end{bmatrix}, \ if \ v \in V_k \\ \begin{bmatrix} k+1 \\ k \end{bmatrix}, \ if \ v \in V_{k+1} \end{cases}$$

(ii) Let $v \neq w$ and $v \in V_k$ and $w \in V_{k+1}$ (or vice versa). Thus, there isn't 2-path between v and w since the graph Γ is a bipartite graph. Indeed, we have $(A^2)_{v,w} = 0$.
(iii) Let $v \neq w$, $v, w \in V_k$ and $P : vuw$ be a 2-path in the graph Γ. Thus $u \in V_{k+1}$ is a $(k+1)$-subspace of $V(q, n)$ such that it contains both of k-subspaces v and w. Since u must contain the subspace $v + w$, it follows that $dim(v \cap w) = k-1$, and $u = v + w$. In other words there is exactly 1 path of length 2 between v and w if and only if $dim(v \cap w) = k-1$. We now, consider $v, w \in V_k$ as vertices of Grassmann graph $G(q, n, k)$. Let G_k be the adjacency matrix of $G(q, n, k)$, then

$$(A^2)_{v,w} = 1 \text{ if and only if } (G_k)_{v,w} = 1, \text{ and } (A^2)_{v,w} = 0 \text{ if and only if } (G_k)_{v,w} = 0$$

where $v, w \in V_1$ and $v \neq w$.
(iv) We now, consider $v, w \in V_{k+1}$ as vertices of Grassmann graph $G(q, n, k+1)$ with vertex set V_{k+1}. By a similar argument which we saw in (iii), we have the following fact,
There is a path of length two (in the graph) between $v, w \in V_{k+1}$ if and only if $dim(v \cap w) = k$.
Therefore, there is a 2-path between $v, w \in V_{k+1}$ as vertices of Γ if and only if v, w are adjacent as vertices of $G(q, n, k+1)$. Let G_{k+1} be the adjacency matrix of the Grassmann graph $G(q, n, k+1)$, then

$$(A^2)_{v,w} = 1 \text{ if and only if } (G_{k+1})_{v,w} = 1, \text{ and } (A^2)_{v,w} = 0 \text{ if and only if } (G_{k+1})_{v,w} = 0.$$

where $v, w \in V_2$ and $v \neq w$.
By our discussion we deduce that,

$$A^2 = \begin{bmatrix} \begin{bmatrix} n-k \\ 1 \end{bmatrix} I_r + G_k & 0 \\ 0 & \begin{bmatrix} k+1 \\ k \end{bmatrix} I_s + G_{k+1} \end{bmatrix}$$

where $r = \begin{bmatrix} n \\ k \end{bmatrix}$ and $s = \begin{bmatrix} n \\ k+1 \end{bmatrix}$. We now can determine the characteristic polynomial of the matrix A^2.

$$P(A^2) = det(\lambda I - A^2) = det(\lambda I_r - \begin{bmatrix} n-k \\ 1 \end{bmatrix} I_r - G_k)det(\lambda I_s - \begin{bmatrix} k+1 \\ k \end{bmatrix} I_s - G_{k+1}).$$

We know from Theorem 1.4, the spectrum of the Grassmann graph $G(q, n, k)$. We now conclude that the eigenvalues of the matrix A^2 are,
$\theta_j = q^{j+1} \begin{bmatrix} k-j \\ 1 \end{bmatrix} \begin{bmatrix} n-k-j \\ 1 \end{bmatrix} - \begin{bmatrix} j \\ 1 \end{bmatrix} + \begin{bmatrix} n-k \\ 1 \end{bmatrix}$ with multiplicity $f_j = \begin{bmatrix} n \\ j \end{bmatrix} - \begin{bmatrix} n \\ j-1 \end{bmatrix}$, $0 \leq j \leq k$
and
$\gamma_i = q^{i+1} \begin{bmatrix} k+1-i \\ 1 \end{bmatrix} \begin{bmatrix} n-k-1-i \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} + \begin{bmatrix} k+1 \\ k \end{bmatrix}$ with multiplicity $e_i = \begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n \\ i-1 \end{bmatrix}$, $0 \leq i \leq k+1$.

Since the eigenvalues of A^2 are squares of the eigenvalues of A and since $Sp(n, q, k)$ is a bipartite graph, then each eigenvalue of $\Gamma = Sp(n, q, k)$ is of the form,
$\pm\sqrt{\theta_j}$, with multiplicity $f_j = \begin{bmatrix} n \\ j \end{bmatrix} - \begin{bmatrix} n \\ j-1 \end{bmatrix}, 0 \leq j \leq k$,
and
$\pm\sqrt{\gamma_i}$ with multiplicity $e_i = \begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n \\ i-1 \end{bmatrix}$, $0 \leq i \leq k+1$. □

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On the $H-$Regular Covering of Graph

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Abstract

In this paper, by reviewing the concept of covering of graphs, we define H-regular covering of graph and we present some properties of this concept. Also, we extend the action of $\pi_1(G, g)$ on $p^{-1}(g)$ to a new action for graphs. We obtain some property of this action.

Keywords: graph, fundamental group, covering of graph.
Mathematics Subject Classification [2010]: Primary: 57M10, 57M12; Secondary: 57M05

1 Introduction

For every two graphs $\tilde{G} = (V_1, E_1)$ and $G = (V_2, E_2)$, recall that p is a covering map from $\tilde{G}$ to G if $p : V_1 \to V_2$ is a surjection and for each $v \in V_1$, the restriction of p to the neighborhood of v is a bijection onto the neighborhood of $p(v)$ in G. Put otherwise, p maps edges incident to v one-to-one onto edges incident to $p(v)$.
Since π_1 is a (covariant) functor, $\pi_1(p) = p_* : \pi_1(\tilde{G}, \tilde{g}) \to \pi_1(G, g)$ is a homomorphism, where $\tilde{g} \in \tilde{G}$ and $g \in G$. If we study p_*, then we can find more information about p (see [1, 2]). Note that $\pi_1(\tilde{G}, \tilde{g_0})$ acts on $p^{-1}(g_0)$ and this property has interesting results (see [3, 4, 5]).

Definition 1.1. [3, Definition, page 280] Let a group G act on a set Y,and let $y \in Y$. Then the orbit of y is

$$o(y) = \{gy : g \in G\} \subset Y,$$

and the stabilizer of y (also called the isotropy subgroup of y) is

$$G_y = \{g \in G : gy = y\} \subset G.$$

It is easy to see that G_y is a subgroup of G. Note that G acts transitively on Y if and only if $o(y) = Y$ for every $y \in Y$.

Lemma 1.2. *[3, Lemma 10.8] If a group G acts on a set Y and if $y \in Y$, then $|o(y)| = [G : G_y]$. In particular, if G acts transitively, then $|Y| = [G : G_y]$.*

In this paper, we introduce an equivalence relation on $p^{-1}(g_0)$. We show that if $H < \pi_1(\tilde{G}, \tilde{g_0})$, then H acts on this new set.

[1] speaker

2 $H-$regular covering of graph

We recall that $p: \widetilde{G} \to G$ is a regular covering of graph, if $p_*\pi_1(\widetilde{G})$ is a normal subgroup of $\pi_1(G)$.

Definition 2.1. Let $H \leq \pi_1(G, g_0)$, a covering map $p: \widetilde{G} \to G$ is an $H-$regular if $p_*\pi_1(\widetilde{G}, \widetilde{g})$ is a normal subgroup of H, for all $\widetilde{g} \in p^{-1}(g_0)$.

Example 2.2. Every a covering map $p: \widetilde{G} \to G$ is an $\{e\}-$regular .

It is easy to see that every regular covering of graph is an $H-$regular covering of graph (it is enough to consider $H = \pi_1(G, g_0)$), but the converse does not hold.

Definition 2.3. We say that $\widetilde{g} \sim_H \widetilde{y}$ if and only if there is an $h \in H$ such that $h\widetilde{g} = \widetilde{y}$ i.e. $\tilde{h}(1) = \widetilde{y}$ where $\tilde{h}$ is the liffting of h at $\widetilde{g}$. Note that $\sim_H$ is an equivalence relation on $p^{-1}(g_0)$. For every $\widetilde{g_0} \in p^{-1}(g_0)$ the equivalence class $[\widetilde{g_0}]$ under relation $\sim_H$ denoted by $p^{-1}_{H,\widetilde{g_0}}(g_0)$. Infact $p^{-1}_{H,\widetilde{g_0}}(g_0) = \{\tilde{g} \in p^{-1}(g_0) | \widetilde{g} \sim_H \widetilde{g_0}\}$ and $Cov_{H,\widetilde{g_0}}(\widetilde{G}/G) = \{f: \widetilde{G} \to \widetilde{G}$ *is an isomorphism of graph such that* $p \circ f = p$ *and* $f(p^{-1}_{H,\widetilde{g_0}}(g_0)) = p^{-1}_{H,\widetilde{g_0}}(g_0)\}$.

Remark 2.4. If $H = \pi_1(G, g_0)$, then $p^{-1}_{H,\widetilde{g_0}}(g_0) = p^{-1}(g_0)$ and $Cov_{H,\widetilde{g_0}}(\widetilde{G}/G) = Cov(\widetilde{G}/G)$.

By Remark 2.4, The following lemma is an extension of [3, Theorem 10.9].

Lemma 2.5. *Let $p: \widetilde{G} \to G$ be a covering of graph, let $g_0 \in G$, and let $H \leq \pi_1(G, g_0)$.*

1. *H acts transitively on $p^{-1}_{H,\widetilde{g_0}}(g_0)$;*
2. *If $\widetilde{G} \in p^{-1}_{H,\widetilde{g_0}}(g_0)$, then the stabilizer of $\widetilde{G}$ is $p_*\pi_1(\widetilde{G}, \widetilde{g})$;*
3. *$|p^{-1}_{H,\widetilde{g_0}}(g_0)| = [\pi_1(G, g_0) : p_*\pi_1(\widetilde{G}, \widetilde{g_0})]$.*

We need the following lemmas for the next theorem.

Lemma 2.6. *Consider a covering map $p: \widetilde{G} \to G$, let $g_0 \in G, \widetilde{g_0} \in p^{-1}_{H,\widetilde{g_0}}(g_0)$ and let $H \leq \pi_1(G, g_0)$. Then $p_*\pi_1(\widetilde{G}, \widetilde{g_0})$ is a normal subgroup of H if and only if $p_*\pi_1(\widetilde{G}, \widetilde{g_0}) = p_*\pi_1(\widetilde{G}, \widetilde{y})$ for all $\widetilde{y} \in p^{-1}_{H,\widetilde{g_0}}(g_0)$.*

The following lemma is an extension of [3, Theorem 10.18] (see Remark 2.4).

Lemma 2.7. *Let $p: \widetilde{G} \to G$ be a covering of graph and let $g_0 \in G$. Then $Cov_{H,\widetilde{g_0}}(\widetilde{G}/G)$ act transitively on $p^{-1}_{H,\widetilde{g_0}}(g_0)$ if and only if $p_*\pi_1(\widetilde{G}, \widetilde{g_0}) = p_*\pi_1(\widetilde{G}, \widetilde{y})$ for all $\widetilde{y} \in p^{-1}_{H,\widetilde{g_0}}(g_0)$.*

By Lemma 2.6 and Lemma 2.7, we have the following theorem.

Theorem 2.8. *The following conditions are equivalent.*

1. *p is an $H-$regular covering of graph at g_0;*
2. *$Cov_{H,\widetilde{g_0}}(\widetilde{G}/G)$ acts transitive on $p^{-1}_{H,\widetilde{g_0}}(g_0)$, for every $\widetilde{g_0} \in p^{-1}(g_0)$;*
3. *$p_*\pi_1(\widetilde{G}, \widetilde{g_0}) = p_*\pi_1(\widetilde{G}, \widetilde{y})$, for every $\widetilde{g_0} \in p^{-1}(g_0)$ and for every $\widetilde{y} \in p^{-1}_{H,\widetilde{g_0}}(g_0)$.*

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Bounds of unique response strong Roman domination

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Abstract

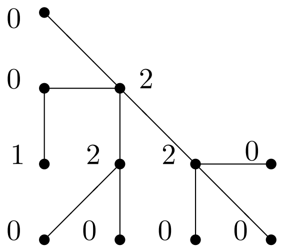
Given a simple graph $G=(V,E)$ with maximum degree Δ. Let (V_0,V_1,V_2) be an ordered partition of V, where $V_i=\{v\in V: f(v)=i\}$ for $i=0,1$ and $V_2=\{v\in V: f(v)\geq 2\}$. A function $f: V\to\{0,1,...,\lceil\frac{\Delta}{2}\rceil+1\}$ is a strong Roman dominating function (StRDF) on G, if every $v\in V_0$ has a neighbor $w\in V_2$ and $f(w)\geq 1+\lceil\frac{1}{2}|N(w)\cap V_0|\rceil$. A function $f: V\to\{0,1,...,\lceil\frac{\Delta}{2}\rceil+1\}$ is a unique response strong Roman function (URStRF), if $w\in V_0$, then $|N(w)\cap V_2|\leq 1$ and $w\in V_1\cup V_2$ implies that $|N(w)\cap V_2|=0$. A function $f: V\to\{0,1,...,\lceil\frac{\Delta}{2}\rceil+1\}$ is a unique response strong Roman dominating function (URStRDF) if it is both URStRF and StRDF. The unique response strong Roman domination number of G, denoted by $u_{StR}(G)$, is the minimum weight of a unique response strong Roman dominating function. In this paper we initiate the study of several mathematical properties of this invariant. We obtain several bounds on such a parameter and give some realizability results for it.

Keywords: Strong Roman dominating function, unique response strong Roman (dominating) function.
Mathematics Subject Classification [2010]: Primary: 05C69

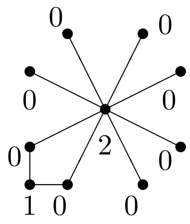
1 Introduction

The history of the Roman Empire gave rise to the mathematical concept of Roman domination, as originally defined and discussed by Stewart [7] in 1999, and ReVelle and Rosing [5] in 2000. The defensive strategy of Roman domination is based on the fact that every place in which there is established a Roman legion (a label 1 in the Roman dominating function) is able to protect itself from external attacks; and that every unsecured (i.e., weak) place (a label 0) must have at least a stronger neighbor (a label 2). In that way, if an unsecured place is attacked, then the stronger neighbor can send it one of the two legions to defend it, (see [2, 3, 4]. Two examples of Roman dominating functions are depicted in Figure 1.
Although these two functions (Figure 1) satisfy the conditions to be Roman dominating functions, they correspond to two very different real situations. The unique strong place 2 in Figure 1 (b) must defend up to 8 unsecured locations from possible external attacks. However, in Figure 1 (a), the task of defending the unsecured locations is divided between several strong locations. These observations have let us to pose the following question: how many weak locations/places can be defended by a strong location occupied by two legions? With this motivation in mind, in [1] Alvarez-Ruiz et al., introduced the concept of strong Roman dominating function.

[1]speaker



(a)



(b)

Figure 1

Let $G = (V, E)$ be a simple graph of *order* $n = |V|$, where $V = V(G)$ and $E = E(G)$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u : uv \in E(G)\}$. If S is a subset of V, then $N(S) = \cup_{x \in S} N(x)$, $N[S] = \cup_{x \in S} N[x]$ and the *subgraph induced* by S in G is denoted $G[S]$. Let E_v be the set of all edges incident with a vertex v in G, that is, $E_v = \{uv \in E(G) : u \in N(v)\}$. The *degree* of a vertex v is $d_G(v) = deg(v) = |E_v|$. The *minimum* and *maximum degree* of G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$. For two vertices u and v in a connected graph G, the distance $d(u, v)$ between u and v is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices of G is the *diameter* of G, which is denoted by diam(G) [8].
For a real-valued function $f : V(G) \to \mathbb{R}$ and $S \subseteq V(G)$, we define $f(S) = \sum_{x \in S} f(x)$. From now on, if $f : V \to \{0, 1, 2, ...\}$ is a function on G, then we let $V_i = \{v \in V : f(v) = i\}$ for $i = 0, 1$ and $V_2 = \{v \in V : f(v) \geq 2\}$. A *strong Roman dominating function* (**StRDF**) on a graph G is a function $f : V \to \{0, 1, ..., \lceil \frac{\Delta}{2} \rceil + 1\}$ such that if $v \in V_0$ for some $v \in V$, then there exists a vertex $w \in N(v)$ such that $w \in V_2$ and $f(w) \geq 1 + \lceil \frac{1}{2} |N(w) \cap V_0| \rceil$. The minimum weight over all strong Roman dominating functions on G is called the *strong Roman domination number* of G, denoted by $\gamma_{StR}(G)$. An *independent strong Roman dominating function* (**IStRDF**) of G is an StRDF such that the set of all vertices assigned positive values is independent. The *independent strong Roman domination number* $i_{StR}(G)$ is the minimum weight of an IStRDF of G. an StRDF of minimum weight is called a $\gamma_{StR}(G)$-function and likewise $i_{StR}(G)$-function is defined. An example of an StRDF and an IStRDF can be seen on the graph in Figure 1 (b), by assigning a 5 to the vertex of maximum degree, a 1 to the vertex of degree 2 and a 0 to the remaining vertices.

In [6], Rubalcaba and Slater studied Roman domination influence of parameters in which the interest is in dominating each vertex exactly once. The authors [6] also introduced the concept of unique response Roman functions which we will adapt the definition for strong Roman functions as follows: *A function $f : V \to \{0, 1, ..., \lceil \frac{\Delta}{2} \rceil + 1\}$ with the ordered partition (V_0, V_1, V_2) of V is a unique response strong Roman function if $w \in V_0$ then $|N(w) \cap V_2| \leq 1$ and $w \in V_1 \cup V_2$ implies that $|N(w) \cap V_2| = 0$. A function $f : V \to \{0, 1, ..., \lceil \frac{\Delta}{2} \rceil + 1\}$, is a unique response strong Roman dominating function, or just URStRDF, if it is a unique response strong Roman function and a strong Roman dominating function. The unique response strong Roman domination number, denoted by $u_{StR}(G)$, is the minimum weight of a URStRDF of G.*

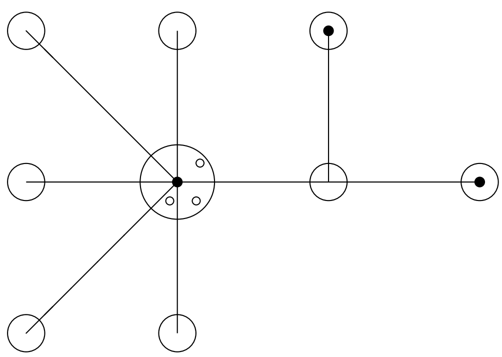


Figure 1

It is worth mentioning that every graph has a unique response strong Roman dominating function since $(\emptyset, V(G), \emptyset)$ is such a function. Moreover, if $f = (V_0, V_1, V_2)$ is a *URStRDF* on G, then V_2 is a 2-packing set. In Figure 3, the black shaded pebble represents a stationary army and the white shaded pebble represents a traveling army. It is easy to check that an attack on any weak vertex of the graph will have three traveling army responding to the attacks.

2 Main Results

We provide in this section some upper and lower bounds for the unique response strong Roman domination number of a graph G in terms of maximum degree, minimum degree, the domination number, the diameter and the order of G. Obviously, every graph of order n, $u_{StR}(G) \leq n$, with equality if and only if each component of G has order at most two. Our next result improves the previous upper bound.

Theorem 2.1. *Let $f = (V_0, V_1, V_2)$ be an StRDF of graph G. If V_2 is independent and no edge of G joins V_1 and V_2, then there is an IStRDF g of G such that $\omega(g) \leq \omega(f)$.*

Theorem 2.2. *If G has an $i_{StR}(G)$-function $f = (V_0, V_1, V_2)$ such that $N(x) \cap N(y) = \emptyset$ for any pair $x, y \in V_2$, then $u_{StR}(G) = i_{StR}(G)$.*

The next result shows that the difference between $u_{StR}(G)$ and $\gamma_{StR}(G)$ can be arbitrarily large.

Proposition 2.3. *For every integer $k \geq 2$, there is a graph G such that $u_{StR}(G) - \gamma_{StR}(G) = k$.*

Theorem 2.4. *For any graph G of order n, $u_{StR}(G) \leq n - \lfloor \frac{\Delta}{2} \rfloor$, and furthermore, this bound is sharp for all graphs of order n with $\Delta(G) = n - 1$.*

Theorem 2.5. *Let G be a connected graph of order n with $\Delta(G) \geq 1$. Then $u_{StR}(G) = \gamma(G) + \lceil \frac{\Delta}{2} \rceil$ if and only if G has a vertex of degree $n - \gamma(G)$.*

Theorem 2.6. *Let G be a graph of order n. If G has a u_{StR}-function $f = (V_0, V_1, V_2, ..., V_{\lceil \frac{\Delta}{2} \rceil + 1})$ such that $V_1 = \emptyset$, then $u_{StR}(G) \leq \frac{(\lceil \frac{\Delta}{2} \rceil + 1)n}{\Delta + 1}$, and this bound is sharp.*

Proof. Let $f = (V_0, V_1, V_2, ..., V_{\lceil \frac{\Delta}{2} \rceil + 1})$ be a u_{StR}-function of G with $V_1 = \emptyset$. Then for every vertex $v \in V - V_0$, we have $f(x) \in V_i$ for $i \geq 2$. Hence, if we have $x \in V$ such that $deg(x) = \Delta$, we may at most assign label $\lceil \frac{\Delta}{2} \rceil + 1$ to the vertex x and label 0 to its neighbors. Thus we have $f(N[x]) = \lceil \frac{\Delta}{2} \rceil + 1$. Therefore we conclude $u_{StR}(G) \leq \frac{\lceil \frac{\Delta}{2} \rceil + 1}{\Delta + 1} n$. The sharpness of the bound may be seen for the star $K_{1,n-1}$ where $n \geq 2$. Clearly, $u_{StR}(G) = \lceil \frac{n-1}{2} \rceil + 1 = \frac{\lceil \frac{\Delta}{2} \rceil + 1}{\Delta + 1} n$. Also the sharpness of the bound may be seen for the other family of graphs as follows. Let G_1 and G_2 be stars $K_{1,2m}$ where $m \geq 1$ such that G obtain from G_1 and G_2 where one of leaves of G_1 is adjacent to one of leaves of G_2. Then $n = |V(G)| = 4m + 2$, $u_{StR}(G) = 2(m + 1)$ and $\Delta(G) = 2m$. Therefore $u_{StR}(G) = 2(m + 1) = \frac{(\lceil \frac{\Delta}{2} \rceil + 1)n}{\Delta + 1}$. □

Theorem 2.7. *Let G be a graph of order n with $\Delta \geq 1$. If $k = \min\{f(v) : f = (V_0, V_1, V_2, ..., V_{\lceil \frac{\Delta}{2} \rceil + 1})$ is a u_{StR}-function and $v \in V - (V_0 \cup V_1)\}$, then $u_{StR}(G) \geq \frac{kn}{\Delta + 1}$, and this bound is sharp*

Proof. Let $f = (V_0, V_1, V_2, ..., V_{\lceil \frac{\Delta}{2} \rceil + 1})$ be a u_{StR}-function of G. Since $|V_0| \leq \Delta(|V_k| + ... + |V_{\lceil \frac{\Delta}{2} \rceil + 1}|)$ and $|V_1| \leq \frac{\Delta}{k}|V_1|$, we have

$$
\begin{aligned}
kn &= k(|V_0| + |V_1| + |V_k| + ... + |V_{\lceil \frac{\Delta}{2} \rceil + 1}|) \\
&\leq k(\Delta|V_k| + ... + \Delta|V_{\lceil \frac{\Delta}{2} \rceil + 1}| + \frac{\Delta}{k}|V_1| + |V_k| + ... + |V_{\lceil \frac{\Delta}{2} \rceil + 1}|) \\
&= \Delta|V_1| + (\Delta + 1)(k|V_k| + ... + k|V_{\lceil \frac{\Delta}{2} \rceil + 1}|) \\
&= (\Delta + 1)(\frac{\Delta}{\Delta + 1}|V_1| + k|V_k| + ... + k|V_{\lceil \frac{\Delta}{2} \rceil + 1}|).
\end{aligned}
$$

Moreover, since $\frac{\Delta}{\Delta+1}|V_1| + k|V_k| + ... + k|V_{\lceil \frac{\Delta}{2} \rceil+1}| \leq u_{StR}(G)$, we deduce that $kn \leq (\Delta + 1)u_{StR}(G)$ yielding the desired bound. The sharpness of the bound may be seen for the star $K_{1,n-1}$ with $n \geq 2$. Clearly, $k = \lceil \frac{n-1}{2} \rceil + 1$ and $u_{StR}(G) = \lceil \frac{n-1}{2} \rceil + 1 = \frac{kn}{\Delta+1}$. □

Theorem 2.8. *Let G be a connected graph with* $\text{diam}(G) \geq 3$*, then*

$$u_{StR}(G) \leq n - \lfloor \frac{\text{diam}(G) - 1}{3} \rfloor.$$

Furthermore, this bound is sharp for paths P_{3k+2} with $k \geq 0$.

Proof. Let $\text{diam}(G) = d = 3m+t$ for some integers $m \geq 1$ and $t \in \{0, 1, 2\}$. Let $P = y_0y_1...y_d$ be a diametral path in G, and let $f : V(P) \to \{0, 1, 2\}$ be a URStRDF defined on P by assigning a 2 to every vertex in $V_2 = \{y_0, y_3, ..., y_{3m}\}$, a 0 to $N(V_2)$ and a 1 to the remaining vertices of P. Note that V_2 is a 2-packing set of P as well as of G. Define now a function $g : V \to \{0, 1, ..., \lceil \frac{\Delta}{2} \rceil + 1\}$ by $g(x) = f(x)$ for $x \in V(P)$ and $g(x) = 1$ otherwise. We also define a function $h : V \to \{0, 1, ..., \lceil \frac{\Delta}{2} \rceil + 1\}$ by $h(y_i) = \lceil \frac{d_G(v_i)}{2} \rceil + 1$ for every $i \in \{0, 3, ..., 3m\}$, $h(x) = 0$ for every $x \in N(y_i)$ such that $i \in \{0, 3, ..., 3m\}$ and $h(y) = g(y)$ for any remaining vertex y. **Clearly,** h **is a URStRDF on** G and we have $\omega(h) \leq \omega(g) \leq f(P) + n - (Diam(G) + 1)$. Thus $\omega(h) \leq \frac{2(Diam(G)+1)+2}{3} + n - Diam(G) - 1$. Finally, by simple calculations we have $u_{StR}(G) \leq \omega(h) \leq \frac{3n-\text{diam}(G)+1}{3} = n - \frac{\text{diam}(G)-1}{3} \leq n - \lfloor \frac{\text{diam}(G)-1}{3} \rfloor$.
For sharpness, let G be a path P_{3k+2} with $k \geq 0$. Then $u_{StR}(G) = \gamma_{StR}(G) = \lceil \frac{2n}{3} \rceil = \lceil \frac{2(3k+2)}{3} \rceil = 2k+2$. On the other hand, we have $n = 3k+2$, $\text{diam}(G) = 3k+1$. Thus, $n - \lfloor \frac{\text{diam}(G)-1}{3} \rfloor = 3k+2 - \lfloor \frac{3k+1-1}{3} \rfloor = 2k+2$. □

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Finding fullerene graphs in group-graphs

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Abstract

A fullerene F is a $3-$connected cubic planar graph with entirely 12 pentagonal faces and $n/2-10$ hexagonal faces where n is the number of vertices of F. It is proved that among the fullerenes graphs only F_{60} is the Cayley graph. Therefore, it is important to investigate this problem for new graph groups. The aim of this paper is to find the structure of fullerenes in some graphs defined by groups.

Keywords: Cayley graph, fullerene, power graph, commuting graph.
Mathematics Subject Classification [2010]: Primary: 22D15, 43A10; Secondary: 43A20, 46H25

1 Introduction

A graph is a pair such as $\Gamma=(V,E)$, where V is the set of vertices and E is the relation between vertices, called the set of edges. A graph without of loop and at most one edge between its two distinct vertices is called a simple graph. The degree of vertex u of the graph Γ is equal to the number of edges on which vertex u is located and is denoted by $d_\Gamma(u)$. The graph Γ, where all degree vertices are the same, is called a regular graph. If all vertices are k, the graph is called $k-regular$. The simple graph that exists an edge between two distinct vertices is called a complete graph. Moreover, a complete graph of order n is denoted by K_n. The complement graph Γ, represented by $\overline{\Gamma}$, is a graph whose vertices are identical to Γ and two vertices in $\overline{\Gamma}$ are adjacent if only if not adjacent in Γ. A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. The planar graph divides the page into a number of enclosed areas, and we call the boundaries of these areas the internal faces of the graph Γ. The whole page is also an expression except the graph Γ, which we call the infinite or external Γ. We show the total number faces of planar graphs Γ with f_Γ. The following theorem, known as Euler's formula, states a simple relation between the number of vertices, edges, and faces of a planar and connected graph and Γ.

Theorem 1.1. ([5]) Suppose Γ is a simple, planar connected graph with n vertices, m edges and f_Γ faces then

$$n-m+f_\Gamma=2.$$

Since the most important issue of finite group theory, namely the classification of finite simple groups, was completed in 1979, one of the major issues of interest to group theory scholars has been the classification of groups with a particular property. The arbitrary group G is classified by the property ρ whenever the group G is uniquely the only group that applies to the property ρ. The problem of classifying groups using commuting graphs and power graphs is an example of such research problems in recent years. For further study on the structure of the commuting graph refer to [1] and the power graphs see [4]. One of the most important graphs that are defined using a group is Cayley graphs.

[1]speaker

Definition 1.2. Take any group G along with a generating set S. We define the Cayley graph $\Gamma(G, S)$ associated to G as the following directed graph:

1. Vertices: the vertices of Γ_G are precisely the elements of G.

2. Edges: for two vertices x, y create the oriented edge (x, y) if and only if there is some generator $s \in S$ such that $x.s = y$. If this happens, we decorate the edge (x, y) with this generator s, so that we can keep track of how we have formed our connections.

From property (1) it follows that if g is connected to h then h is also connected to g, so the graph is well defined. Property (2) concludes that the Cayley graph is simple. See [2] for more on Cayley graphs.

A Fullerene is a molecule composed of carbon atoms in a closed form. The graphs of these molecules are $3-$ regular, $3-$connected , planar and their faces are r edges and 6 edges, where r can be one of the numbers 3, 4, 5. This molecular graphs are also known as $[3, 6]$-fullerene, $[4, 6]$-fullerene and $[5, 6]$-fullerenes. For example, the $[5, 6]$-fullerene graph is a planar, 3-regular graph whose faces are a combination of hexagons and pentagons. Such graphs include n vertices and $3n/2$ edges. We show $[5, 6]$ -fullerene with n vertices by F_n. Using the Euler theorem, it is simply proved that these fullerenes comprise twelve pentagonal and $n/2 - 10$ hexagonal rings. Figure 1 shows the fullerene graph F_{20}, which is a regular twelve-dimensional graph with no hexagon.

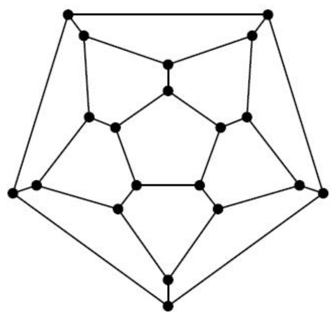

Figure 1: Fullerene graph F_{20}

2 Main Results

In the article [3], it is proved that among the fullerenes graphs only F_{60} is the Cayley graph. Therefore, it is important to investigate this problem for new graph groups. See [2, 3] for more on Cayley graphs and Fulleren graphs. The aim in this section is to find the structure of fullerenes in some graphs defined by groups.

Theorem 2.1. *([3]) The fullerene graph F_{60} is a Cayley graph.*

For a finite group G and its subset X, the commuting graph $C(G, X)$ is a graph with the set of vertices X such that for every $x, y\ inX$, xy is an edge if and only if $xy = yx$. This graph has been investigated in various ways. We consider this article by considering $Z(G) = \{x \in G | xy = yx, \forall y \in G\}$. We consider the two states $X = G$ and $X = G \setminus Z(G)$, and show theses graphs by $\Delta(G)$ and $\Gamma(G)$, respectively. Note that the commuting graph $\Gamma(G)$ has $\mid G \mid - \mid Z(G) \mid$ vertices. Also the degree of any vertex x in $\Gamma(G)$ is $|C_G(x)| - |Z(G)| - 1$, where $C_G(x) = \{g \in G \mid xg = gx, \forall x \in X\}$ is the centralizer of the x in G.

Theorem 2.2. *Let G be a group. The commuting graph $\Gamma(G)$ cannot be a fullerene graph.*

In [1] the planar noncommuting graph is examined to assume G is a non-abelian group. The graph $\bar{\Gamma}(G)$ is planar if and only if G is isomorphism with one of the groups D_8, Q_8 or S_3 . Now using this, we prove the following theorem.

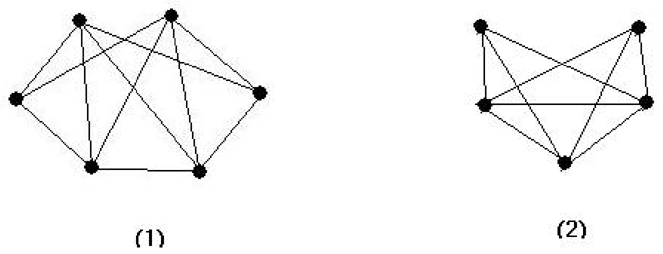

Figure 2: (1) The noncommting $\bar{\Gamma}(D_8) \cong \bar{\Gamma}(Q_8)$ (2) The noncommuting graph $\bar{\Gamma}(S_3)$.

Theorem 2.3. *Suppose G is a nonabelian group. The noncommuting graph $\bar{\Gamma}(G)$ cannot be a fullerene graph.*

We present the power graph $P(G)$ of a finite group G, $V(P(G)) = G$ and two distinct vertices x and y are adjacent in $P(G)$ if and only if one can be a power of the other. The power graph $P(G)$ and the commuting graph with the vertex set G, that we show with $\Delta(G)$ are not fullerene graph.

Theorem 2.4. *The only fullerene graph among power graphs and commuting graphs are $P(Z_4)$ and $\Delta(Z_4)$.*

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Efficient algorithms for sorting geometrical shapes using trees

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Abstract

This paper proposes two new and efficient ways for sorting and querying simple geometrical shapes such as intervals and circles. In both algorithms, we first introduce a map that converts each interval or a circle into a point by adding one dimension. Then, an algorithm is presented for sorting intervals based on the quad-tree and a new tree is introduced for searching circles. The advantage of the presented methods in comparison with the conventional methods is that to build a tree, it is not necessary all objects have been defined at the first. With the article methods we can edit in the tree during the execution processes.

Keywords: Interval sorting; Interval query; Quad-tree; Circle query.

Mathematics Subject Classification [2010]: Primary: 68R10; Secondary: 68P10

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1 Introduction

In computer science, a sorting algorithm is an algorithm that puts elements of a list in a certain order [1]. Efficient sorting is important for optimizing the efficiency of other algorithms (such as search and merge algorithms) that require input data to be in sorted lists. The problem of sorting includes different kinds of it, from the sorting a sequence of real numbers until the sorting of geometrical shapes such as intervals and circles.
The intervals sorting are done in different positions such as the project management timelines, sound and video editions, and etc [3, 4]. The trivial solution for interval query is to visit each interval and test whether it intersects the given interval, which requires $O(n)$ time, where n is the number of intervals in the collection. On the other hand, searching a circle among a multitude of two-dimensional circles is required and it is clear that to add or search a circle, one should not explore all the circles. So the tree-based methods were further developed. The most important reason for choosing trees is their high screening ability. The aim of this article is to sort the simple geometrical shapes by using trees with more than two children nodes. In order to sort the intervals, we use the so known quad-tree and for the second case, we introduce an inventive tree that has shown good efficiency in inserting nodes and searching.

2 Main Results

In this section, we first review the quad-trees and describe a method for sorting and querying the intervals by using these trees. Then, we introduce a new six-nodes tree for sorting and collision detection of circles.

[1] speaker

2.1 The quad-tree

A quad-tree is a tree data structure in which each internal node has exactly four children. Quad-trees are the two-dimensional analog of oct-trees and are most often used to partition a two-dimensional space by recursively subdividing it into four quadrants or regions [2]. In general, if we want to add two independent coordinates such as (p, q) as a node to the quad-tree, we will have a point with these coordinates on the XY-plane and we imagine a Cartesian local coordinate system on it which divides the region into four equal quadrants.

2.2 The new interval sorting algorithm

In this section we describe sorting and effectively searching algorithms for one dimensional intervals. These algorithms can be extended for other regular shapes such as squares with their special conditions.
In the proposed algorithm the corresponding tree from the set of intervals S, can be built as follows. Firstly, we convert each interval in S into equivalent point in two-dimensional space by adding one dimension. Suppose that $I = [a, b] \in S$ be a real interval with the radius $x = (a + b)/2$ and the center $r = (b - a)/2$. We suppose a coordinate system in plane with X-axis for denoting the center of intervals and the R-axis for displaying the radius of intervals. In this system each interval is displayed by a unique ordered pair (x, r). Then we consider an interval with arbitrary nonnegative radius as query interval for finding all intervals that intersect with this interval. We denote this query interval by point $(x_q, -r_q)$. Let $r(x) = |x - x_q| - r_q$. If we denote the space above $r(x)$ by Ω, then for each pair $(x_p, r_p) \in \Omega$, we get

$$|x_p - x_q| - r_q \leq r_p,$$

which is the same condition for intersection of two intervals.
Now, our problem is sorting the equivalent points of intervals. For this purpose, we use quad-tree and in order to increase its efficiency, we rotate each node in the XR-plane counterclockwise through angle 45 degrees about the origin of its local coordinate system. In this case, half of the children nodes of each node will be ignored (see Fig. 1).

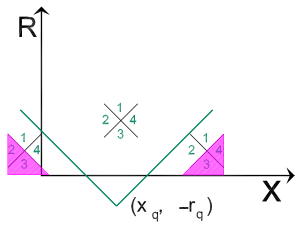

Figure 1: The position of rotated nodes with respect to the region Ω.

Theorem 2.1. *The complete construction operation takes $O(n \log n)$ time, where n is the number of intervals in S.*

Now, the algorithms for construction and search of interval queries are implemented by using quadtrees. The first step is to define a function that returns the appropriate child node index according to the geometric position of the considered point. Then, according to the geometry of the problem (see Fig. 1), the search (query) procedure can also be done.

2.3 The new circle sorting algorithm

In this section we describe the structure and construction of proposed tree structure for sorting and querying 2D circles. This idea can also be extended for other regular shapes such as polygons with their special conditions. Similar to the method presented for interval searching, we will convert each circle into a unique tree-dimensional point in the conventional space XYR, where the X and Y axes are used to represent the coordinates of the center and the R-axis is used to show the radius of the circle. Now we consider a circle with arbitrary nonnegative radius r_q, for finding all circles that have collision with this circle. We denote this query circle by point $(x_q, y_q, -r_q)$ and imagine a standard test cone whit its vertex at this point. With the above assumptions, a point P is on or inside the test cone, if and only if the relation

$$\left((x_P - x_Q)^2 + (y_P - y_Q)^2\right)^{1/2} \leq r_P + r_Q,$$

satisfies, which is the same codition for collision of two circles. At this stage we need to sort the converted points properly and effectively. For this purpose, we introduce a new six-nodes tree, which we call it as "hexatree" (see Fig. 2). The main reason for choosing this new tree is that we need a partition that matches

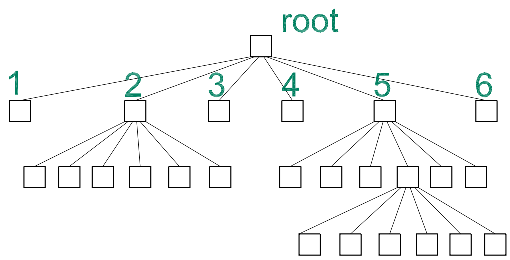

Figure 2: A sample of the haxatree.

with the standard cone, especially with the slope of standard cone. Also in tree traversing, as few nodes as possible should be selected through the nodes according to their position in the converted environment (compared to the test cone) to continue the search path. So the best partitioning is obtained by considering the position of each point as the center of a cube. By connecting this point with the vertices of each side of the cube, we obtain six pyramids. Since we are only dealing with four pages connected to the center of the cube, in practice, we consider the base of each pyramid to be infinite. In this way, the 3D space is divided into the completely separated partitions without a bulky intersection.

The process for adding a new node (circle) in the tree is as follows: It starts from the root node and then finds the first step of the path and goes to the appropriate child node. If it is empty, it is added there. Otherwise, according to its position relative to the middle node, it goes to the appropriate child node. This process continues until the node is added that requires a simple function to determine the index of the new node in the tree that depends on the geometric position of the input argument. The cost of operation of this part is logarithmic and it is done quickly. For the search algorithm, we define a function to investigate the collision of two standard cones.

An implementation of the proposed algorithm has been shown in Fig. 3, for collision detection of a query circle (in red) with a set of 1000 circles (in black) with random radii in $[10, 50]$ and centers in $[1, 1000] \times [1, 1000]$. The total number of visited circles (in green) is 375, while the number of answer circles (in yellow) is 9.

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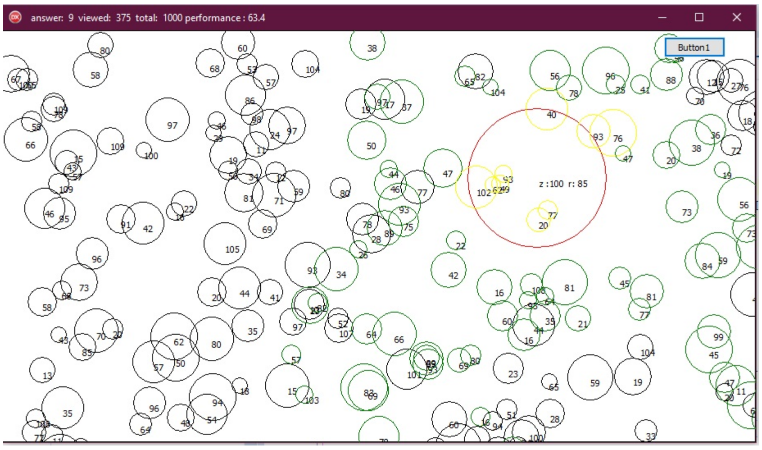

Figure 3: Collision detection of a query circle by proposed method.

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Block designs admitting classical groups as flag-transitive automorphism groups

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Abstract

In this talk, we give a survey on a classification of 2-designs with $\gcd(r, \lambda) = 1$ admitting a flag-transitive automorphism group G whose socle X is a finite simple classical group of Lie type.

Keywords: 2-design, flag-transitive, automorphism group, primitive group.

Mathematics Subject Classification [2010]: Primary: 05B05 , 20B25; Secondary: 05B25

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1 Introduction

A 2-(v, k, λ) design $\mathcal{D}$ is a pair $(\mathcal{P}, \mathcal{B})$ with a set $\mathcal{P}$ of v points and a set $\mathcal{B}$ of b blocks such that each block is a k-subset of $\mathcal{P}$ and each two distinct points are contained in λ blocks. The *replication number* r of $\mathcal{D}$ is the number of blocks incident with a given point. We shall always assume that $\mathcal{D}$ is nontrivial, that is $2 < k < v$. If $b = v$ (or equivalently $r = k$), the design $\mathcal{D}$ is called *symmetric*. So for a nonsymmetric design, we have $b > v$ and $r > k$. An *automorphism* of $\mathcal{D}$ is a permutation on $\mathcal{P}$ which maps blocks to blocks and preserving the incidence. The *full automorphism* group $\mathrm{Aut}(\mathcal{D})$ of $\mathcal{D}$ is the group consisting of all automorphisms of $\mathcal{D}$. A *flag* of $\mathcal{D}$ is a point-block pair (α, B) such that $\alpha \in B$. For $G \leqslant \mathrm{Aut}(\mathcal{D})$, G is called *flag-transitive* if G acts transitively on the set of flags. The group G is said to be *point-primitive* if G acts primitively on $\mathcal{P}$. A group G is said to be *almost simple* with socle X if $X \trianglelefteq G \leqslant \mathrm{Aut}(X)$, where X is a nonabelian simple group. For finite simple classical groups of Lie type, we adopt the standard notation as in [10]. Further notation and definitions in both design theory and group theory are standard and can be found, for example, in [6, 7, 11, 12].

2 Main Results

The main aim of this talk is to study 2-designs with flag-transitive automorphism groups. In 1988, Zieschang [19] proved that if an automorphism group G of a 2-design with $\gcd(r, \lambda) = 1$ is flag-transitive, then G is a point-primitive group of affine or almost simple type. Such designs admitting an almost simple automorphism group with socle being an alternating group, a sporadic simple group, a projective special unitary group or a finite simple exceptional group have been studied in [1, 3, 14, 15, 17, 18]. The present talk is devoted to determining all possible 2-designs with $\gcd(r, \lambda) = 1$ admitting a flag-transitive almost simple automorphism group G with socle X being a finite simple classical group of Lie type. Our main result is Theorem 2.1 below:

[1] speaker

[2] The main results presented in this talk is part a joint work with Seyed Hassan Alavi and Ashraf Daneshkhah, Department of Mathematics, Faculty of Science, University of Bu-Ali Sina, Hamedan, Iran.

Theorem 2.1. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial 2-(v, k, λ) design with r being coprime to λ, and let α be a point of $\mathcal{D}$. If G is a flag-transitive automorphism group of $\mathcal{D}$ with socle X being a finite simple classical group of Lie type and $H = G_\alpha$, then $\mathcal{D}$ belongs to one of the eight infinite families of 2-designs, or (v, b, r, k, λ), G and G_α are as in lines* 1-6 *of* Table 1.

Table 1: Some nontrivial 2-design $\mathcal{D}$ with $\gcd(r, \lambda) = 1$ admitting flag-transitive and point-primitive automorphism group G.

Line	v	b	r	k	λ	G	G_α	G_B	Aut($\mathcal{D}$)	Design	References
1	6	10	5	3	2	$PSL_2(5)$	D_{10}	Sym_3	$PSL_2(5)$		[5, 16, 17]
2	7	7	3	3	1	$PSL_2(7)$	Sym_4	Sym_4	$PSL_2(7)$	$PG_2(2)$	[2, 5, 8, 13]
3	8	14	7	4	3	$PSL_2(7)$	7:3	Alt_4	$2^3{:}PSL_2(7)$		[3]
4	28	36	9	7	2	$PSL_2(8)$	D_{18}	D_{14}	$PSL_2(8){:}3$		[5, 16]
5	10	15	9	6	5	$PSL_2(9)$	$3^2{:}4$	Sym_4	Sym_6		[16, 17]
6	11	11	5	5	2	$PSL_2(11)$	$PSL_2(5)$	$PSL_2(5)$	$PSL_2(11)$	Hadamard	[2, 8, 13]

Note: The subgroups G_α and G_B are point-stabiliser and block-stabiliser, respectively.

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A note on Plesnik's long-open standing problem

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Abstract

The classical papers of Plesnik (1975) and (1984) are two famous manuscripts in metric graph theory in which some pretty and long-standing problems were posed. Plesnik determined some sharp lower and upper bounds for an average distance of graph and digraph with given order and diameter. In these articles, he addressed some problems on the average distance of graphs which remains unsolved for a long time. One of the most well-known of these problems is as follows:

Problem [Plesnik (1975)]: What is the maximum average distance among all graphs of given order and diameter?

The main contribution of this paper is to investigate the long-standing open problem posed by Plesnik in 1975. We prove that if T is a tree with a maximum average distance of given order and diameter then T is a centered tree such thatthe branches of the center of T are almost isomorphic (differ by at most one pendant vertex at the last level) and the degrees of vertices except the center are at most three. Moreover, the leaves are at the last level that is, the distance r from the center. Furthermore, we illustrate a class of trees with a maximum average distance of given order and diameter such that the obvious bound $\mu(G) \leq d$ is best possible.

Keywords: average distance, Djokovic-Winkler's relation, cut method, tree, diameter.
Mathematics Subject Classification [2010]: Primary: 05C12, 05C69; Secondary: 91A43

1 Introduction

Average distance (alias Wiener index or total distance) is one of the most studied graph invariants in mathematics. Since distance is one of the main bases of science, average distance has been applied in other sciences extensively [1]. The classical papers of Plesnik [2, 3] are two famous manuscripts in which some pretty and long-standing problems in metric graph theory were posed. Plesnik determined some sharp lower and upper bounds for an average distance of graph and digraph with given order and diameter [3]. In [2], he previously addressed a problem on the average distance of graphs which remains unsolved for a long time. In the following, we mention this problem with some related work and conjectures.

Problem 1 [2]**:** What is the maximum average distance among all graphs of given order and diameter?

In [4], Wang et al. solved a special case of the above problem when G is a caterpillar tree of given order and diameter. Mukwembi and Vetrik in [5] considered the trees of diameter at most 6 and found out some sharp bounds for the maximum average distance. Recently, Sun et al. [6] investigated maximum average distance for graphs of order n with a diameter smaller than or equal to 4 and also characterize the graphs with a diameter greater than $n - c$ when $1 \leq c \leq 4$.

[1]speaker

It is well-known that for each connected graph G of radius r and diameter d we have $r \leq d \leq 2r$. As it was hinted in Plesnik papers, it is natural to consider the same problem for the radius of graphs. The next open problem that is so close to Problem 1 is as follows:

Problem 2 [7]**:** What is the maximum average distance among all graphs of given order and radius?

Chen et al. [7] solved Problem 2 when G is a graph of order n and radius 2. Recently, Nadjafi-Arani with Das found a sharp bound on maximum average distance of trees and graphs with given radius [8].

The first natural question that comes to mind is that why we can find a large number of articles in relation to (extermal) average distance and other graph invariants such as minimum degree, independent number and so on. But there are just a small number of researches about the old problems 1 and 2. On the other hand, available articles just consider so special cases of problems. In fact, although these problems have a simple appearance, it seems to be quite challenging even for some special classes of graphs. To see how hard Problem 1 is, consider the next Graffiti conjecture that is a much-restricted case of Problem 1.
conjecture 1 [9, Conjecture 7]**:** Let G be a graph with diameter $d > 2$ and order $2d + 1$. Then

$$\mu(G) \leq \mu(C_{2d+1})$$

where C_{2d+1} denotes the cycle of length $2d + 1$ and $\mu(G)$ shows the average distance of graph G.

To the best author's knowledge, there is no more researches about the maximum average distance with given order and diameter/radius except the mentioned-above articles. It can also be another reason to show that we encounter a challenging problem.

The cut method that was extensively studied in metric graph theory is one of the new methods for computing the average distance of graphs. The method has its roots in the embedding of graphs into hypercubes (see [1]). Recently, the author with S. Klavžar proved that it is possible to apply the cut method for weighted graphs [10]. These extensions are called the extended cut method and play the main role in our context.

In this paper, we apply Djokovic-Winkler's relation to express the average distance , $\mu(G)$, of a graph G in terms of its canonical metric representation. Then, by using the extended cut method and optimization techniques, determine the structure of trees with the maximum average distance of given order and diameter. Since every connected graph has a spanning tree of the same radius then we extend our results to general graphs to characterize the graphs with the maximum average distance of given order and radius. At the end, we apply our main theorems to introduce a class of balanced star trees , $T(n, d)$, of given order n and diameter d such that the obvious bound $\mu(G) \leq d$ is the best possible that is, $\lim_{n\to\infty} \mu(T(n, d)) = d$. Also, we illustrate that there are some counterexamples to show that although the class of trees $T(n, d)$ is best possible, there are other graphs with more average distances.

2 Preliminaries

Let $G = (V(G), E(G))$ be a connected graph of order n and size m. We consider $d_G(u, v)$ the shortest path distance between u and v of a graph G and simplify the notation to $d(u, v)$ when the graph is clear. The *Wiener index* or *total distance* $W(G)$, is the sum of distances between all pairs of vertices of G. As a very similar definition of Wiener index the *average distance* $\mu(G)$ defines $\mu(G) = \frac{2}{n(n-1)}W(G)$ [11, 12]. The *eccentricity* of a vertex v in a connected graph G is the maximum distance between v and any other vertex u of G which is shown by $\epsilon(v)$. The *radius* of G, $r(G)$ (for short r) is the minimum eccentricity of graph vertex and the *diameter* of G, $d(G)$ (for short d) is defined as follows:

$$d(G) = \max\{\epsilon(v) \,|\, v \in V(G)\}.$$

Let $deg(u)$ and $N(u)$ show degree of u and the set of vertices adjacent to u respectively. A *spanning tree* T of a connected undirected graph G is a treeincludes all of the vertices and some or all of edges of G. It is well-known that any graph G of order n and radius r contains a spanning tree with the same radius. The *center (s)* of a graph is(are) the set of all vertices of minimum eccentricity. A tree has precisely one center or two centers called centered or bicentered tree respectively. In a bicentered tree, both of two centers are adjacent and created an edge that is called *edge center*. A *branch* of a vertex v at a tree T is a maximal connected subtree of T in which v is a pendant vertex of it.

For a graph G, the Djoković-Winkler's relation Θ [13, 14] is defined on $E(G)$ as follows:

Let $e = xy$ and $f = uv$ be two arbitrary edges of G, then $e\Theta f$ if

$$d(x, u) + d(y, v) \neq d(x, v) + d(y, u).$$

Relation Θ is reflexive and symmetric and its transitive closure Θ^* is an equivalence relation. The partition of $E(G)$ induced by Θ^* is called the Θ^**-partition* or *cuts*. Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ denotes the Θ^*-partition of $E(G)$. A partition $\mathcal{E} = \{E_1, \ldots, E_r\}$ of $E(G)$ is *coarser than* $\mathcal{F}$ if each set of E_i be the union of one or more Θ^*-classes of G.

A *vertex weighted graph* (G, w) is a graph $G = (V(G), E(G))$ together with the weight function $w : V(G) \to \mathbb{R}^+$. The Wiener index $W(G, w)$ of (G, w) is then defined as [15]:

$$W(G, w) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} w(u)\ w(v)\ d_G(u, v)\,,$$

clearly, if $w \equiv 1$ then $W(G, w) = W(G)$.

The *cut method* is a new method for computing the Wiener index of graphs. In this method, the Wiener index of a connected graph can be expressed as a sum of the Wiener indices of weighted quotient graphs with respect to an arbitrary combination of Θ^*-classes. For details see the survey [16].

3 Main Results

It is well-known that for any connected graph G, $r(G) \leq d(G) \leq 2r(G)$. By considering this close relationship between radius and diameter of graphs, solving anyone of Problems 1 or 2 gives an idea to solve other. In this section, we determine a view of the structure of trees with a maximum average distance of given order n and radius r. Then extend our results to trees of given order and diameter.

Lemma 3.1. *Let T be a tree with maximum average distance of given radius $r \geq 2$ and order $n \geq 2r + 2$, then T is a centered tree.*

Theorem 3.2. *Let G be a graph with maximum average distance of given order n and radius r, then*

1. *If $n = 2r$ or $2r + 1$ then $G \cong P_n$.*
2. *If $r = 1$ then $G \cong S_n$.*
3. *If $r \geq 2$ and $n \geq 2r + 2$ then G is a centered tree such that the branches of the center are almost isomorphic and the degree of vertices except the center are at most three. Moreover, the leaves are at the last level of T that is, the distance r from the center.*

Since each tree T of order n and diameter $d = 2r$ is a centered tree of radius r therefore, similar to the last theorem we have:

Theorem 3.3. *Let T be a tree with maximum average distance of given order $n \geq 2r + 1$ and diameter $d = 2r$, then*

1. *If $n = 2r + 1$ then $G \cong P_n$.*
2. *If $r = 1$ then $G \cong S_n$.*
3. *If $r \geq 2$ and $n \geq 2r + 2$ then G is a centered tree such that the branches of the center are almost isomorphic and the degree of vertices except the center are at most three. Moreover, the leaves are at the last level of T that is, the distance r from the center.*

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The Mathieu simple group as automorphism group of 2-designs

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Abstract

In this talk, present some recent studies on 2-designs with $\lambda^2 \leqslant r$ admitting a flag-transitive automorphism group of almost simple type with Sporadic groups socle [1].

Keywords: Symmetric design, Automorphism group, Flag-transitive, Point-primitive, Symplectic group.
Mathematics Subject Classification [2010]: Primary: 05B05; Secondary: 05B25, 20B25

1 Introduction

A 2-(v, k, λ) design is an incidence structure pair $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is a v-set, called points, and a collection B of size b consisting k-subsets of $\mathcal{P}$, called blocks, such that each element of $\mathcal{P}$ is incident with exactly r blocks and any 2-subset of $\mathcal{P}$ is incident with exactly λ blocks. The integer numbers v, b, r, k and λ are called the parameters of the design. We shall always assume that $\mathcal{D}$ is non-trivial, that is, $2 < k < v$. If $b \neq v$ (or equivalently $r \neq k$), the design $\mathcal{D}$ is called a non-symmetric design.Two designs $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1)$ and $\mathcal{D}_2 = (\mathcal{P}_2, \mathcal{B}_2)$ are isomorphic, written $\mathcal{D}_1 \cong \mathcal{D}_2$, if there exists a bijection $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ such that $\mathcal{B}_1^\alpha = \mathcal{B}_2$. An automorphism of $\mathcal{D}$ is an isomorphism from $\mathcal{D}$ to itself. The set of all automorphisms of $\mathcal{D}$ forms a group which is denoted by $\mathrm{Aut}(\mathcal{D})$, and is called the full automorphism group of $\mathcal{D}$. Any subgroup of $\mathrm{Aut}(\mathcal{D})$ is called an automorphism group of $\mathcal{D}$. For $G \leqslant \mathrm{Aut}(\mathcal{D})$, the design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is called point-primitive if G acts primitively on $\mathcal{P}$, and flag-transitive if G acts transitively on the set of flags, where a flag of $\mathcal{D}$ is an incident point-block pair (α, B).

2 Main Results

Tian and Zhou in [4], completely classified all flag-transitive, point-primitive 2-(v, k, λ) symmetric designs with sporadic socle. Also in [5, Corollary 2], Zhan and Zhou shown that if a 2-design with $(r, \lambda) = 1$ admits a flag-transitive automorphism group of almost simple type with sporadic socle, then there are only three pairs $(\mathcal{D}, G)$. Furthermore in 2017[6], proved that if $\mathcal{D}$ be a non-trivial non-symmetric 2-(v, k, λ) design with $\lambda \geqslant (r, \lambda)^2$, admitting a flag-transitive automorphism group of almost simple type with sporadic socle, then there are exactly such 33 non-isomorphic designs. In [3], Liang and Zhou proved that if $\mathcal{D}$ is a nontrivial nonsymmetric 2-(v, k, λ) design admitting a flag-transitive point-primitive automorphism group G, then G must be an affine or almost simple group. Moreover, if the socle of G is sporadic, then $\mathcal{D}$ is the unique $2 - (176, 8, 2)$ design with $G = HS$, the Higman-Sims simple group. In [1], we prove that if $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a non-trivial symmetric (v, k, λ) design with $\lambda \geqslant 1$, and G is a flag-transitive and point-primitive automorphism group of $\mathcal{D}$ and λ divides k and $k \geqslant \lambda^2$, then G is an affine or an almost simple type. In [2], we study non-trivial 2-(v, k, λ) design, admitting a flag-transitive and point-primitive automorphism group of almost simple type with socle a sporadic simple group. In this talk, we report our result on Mathieu group.

[1]speaker

Theorem 2.1. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial (v, k, λ) design with $\lambda^2 \leqslant r$, and let G be a flag-transitive and point-primitive automorphism group of $\mathcal{D}$ whose socle is one of the Mathieu simple groups. Then there are* 117 *non-isomorphic designs.*

Acknowledgment

The main results of this talk are part of joint papers with Seyed Hassan Alavi, Ashraf Daneshkhah [1] and [2].

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Edge chromatic vertex stability number

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Abstract

For a graph G, the edge chromatic stability number of graphs, $vs_{\chi'}(G)$, is the minimum number of vertices of G whose removal results in a graph $H \subseteq G$ with $\chi'(H) \neq \chi'(G)$. In this paper, we study this parameter and prove some general results for it and determine $vs_{\chi'}(G)$ for specific classes of graphs.

Keywords: edge chromatic vertex stability number; chromatic index; corona; join.

Mathematics Subject Classification [2010]: Primary: 05C15; Secondary: , 05C25

1 Introduction

Let $G = (V, E)$ be a simple graph with n vertices. Throughout this paper we consider only simple graphs. A graph is empty if $E(G) = \varnothing$. We follow the standard graph notation and definitions [4]. Kemnitz and Marangio in [4], for an arbitrary invariant of G, introduced the $\rho-$edge stability number $es_\rho(G)$ of G as the minimum number of edges of G whose removal results in a graph $H \subseteq G$ with $\rho(H) \neq \rho(G)$ or with $E(H) = \emptyset$. Then they gave some general lower and upper bounds for the ρ-edge stability number. Also they studied the χ'-edge stability number of graphs, where $\chi' = \chi'(G)$ is the chromatic index of G. They proved some general results for the so-called chromatic edge stability index $es_{\chi'}(G)$ and determined $es_{\chi'}(G)$ exactly for specific classes of graphs. For recent results on chromatic edge stability number and stabilizing on the distinguishing number of a graph, see [1, 2, 5].

Motivated by these papers, we state the following definitions:

Definition 1.1. The chromatic vertex stability number $vs_{\chi'}(G)$ of a non-empty graph G, is the minimum number of vertices of G whose removal results in graph $H \subseteq G$ with $\chi'(H) \neq \chi'(G)$. If G is empty, then $vs_{\chi'}(G) = 0$.

In this paper, we study χ'-vertex stability number of graphs, and compute the $vs_{\chi'}(G)$ for specific graphs. Also, we study the χ'-vertex stability number of join and corona product of two graphs.

[1]speaker

2 Main Results

A function $c : E(G) \longrightarrow \{1, ..., k\}$ such that $c(e_1) \neq c(e_2)$ for any two adjacent edges e_1 and e_2 is called a k-edge coloring of G, and G is called k-edge colorable. The minimum k for which G is k-edge colorable is the chromatic index $\chi'(G)$ of G. By Vizing's Theorem, the chromatic index can only attain one of two values, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. Graphs with $\chi'(G) = \Delta(G)$ are called class 1 graphs and graphs with $\chi'(G) = \Delta(G)+1$ are called class 2 graphs. As in [4], we consider the invariant $class(G) = \chi'(G)-\Delta(G)+1 \in \{1, 2\}$. A graph G is called overfull if its order n is odd and if it contains more than $\Delta(G)(n-1)/2$ edges. Obviously, an overfull graph must be a class 2 graph. Note that $\chi'(G)$ is an invariant which is monotone increasing, integer valued, and maxing [4].

Lemma 2.1. *If G is a class 1 graph, then $vs_{\chi'}(G) \geq vs_{\Delta}(G)$.*

Proposition 2.2. *If G is a class 1 graph and there is a vertex set V' such that $\mid V' \mid= vs_{\Delta}(G)$, $\Delta(G-V') < \Delta(G)$ and $(G - V')$ is in the class 1, then $vs_{\chi'}(G) = vs_{\Delta}(G)$.*

Theorem 2.3. *If G is a class 2 graph, then $vs_{\chi'}(G) = min\{vs_{\Delta}(G), vs_{class}(G)\}$.*

Let $t^*(G)$ be the minimum number of edges in a color class of the graph G where the minimum is taken over all edge colorings of G with $\chi'(G)$ colors. If G is nonempty, then removing a vertex of any edges in one color class from G reduces the chromatic index, thus $vs_{\chi'}(G) \leq t^*(G)$ follows.

Theorem 2.4. *(i) There exists a graph G such that $\mid vs_{\chi'}(G) - t^*(G) \mid$ can be arbitrarily large.*

(ii) There exists a graph G such that $|vs_{\chi'}(G) - vs_{\Delta}(G)|$ can be arbitrarily large.

Theorem 2.5. *For every $k \in \mathbb{N}$, there exists a graph G such that $vs_{\chi'}(G) = k$.*

Here, we study the $vs_{\chi'}$ for corona product and join of two graphs. First we consider the corona product of two graphs. We recall that the corona of G and H is denoted by $G \circ H$, is a graph made by a copy of G (which has n vertices) and n copy of H and joining the i-th vertex of G to every vertex in the i-th copy of H. Obviously $\Delta(G \circ H) = \Delta(G) + |V(H)|$.

Proposition 2.6. *If $\Delta(G) \geq 1$, then $vs_{\Delta}(G \circ H) \leq vs_{\Delta}(G)$.*

Theorem 2.7. *Let $\Delta(G) \geq 1$ and V' be the subset of vertices of G which its removing get $vs_{\chi'}(G)$. If $\Delta(G - V') + 1 < \Delta(G)$, then $vs_{\chi'}(G \circ H) = vs_{\chi'}(G)$.*

Theorem 2.8. *If $G \circ H$ is in the class 2, then $vs_{\chi'}(G \circ H) \leq \gamma(V_{\Delta(G)})$.*

The join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv|u \in V(G_1) \text{ and } v \in V(G_2)\}$. Let $n_1 = |V_1|$, $n_2 = |V_2|$, $\Delta_1 = \Delta(G_1)$ and $\Delta_2 = \Delta(G_2)$. Clearly, $n(G_1 \vee G_2) = n_1 + n_2$ and $\Delta(G_1 \vee G_2) = max\{n_1 + \Delta_2, n_2 + \Delta_1\}$.

Proposition 2.9. *$vs_{\Delta}(G_1 \vee G_2) \leq 2$.*

Proposition 2.10. *If $G = G_1 \vee G_2$ and $\Delta_1 + n_2 \neq \Delta_2 + n_1$, then $vs_{\Delta}(G) = 1$.*

Theorem 2.11. *If $G = G_1 \vee G_2$ and $n_1, n_2 \geq 2$, then $vs_{\chi'}(G) \leq 4$.*

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Some properties on the bicolor basis graph of a matroid

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Abstract

Let ϕ be a function which is defined on elements of a matroid M onto $\{1, 2\}$. The bicolor basis graph $G(\mathcal{B}(M), \phi)$, is a graph with vertex set given by the set of bases of M,$\mathcal{B}(M)$, in which two bases B_1 and B_2 are adjacent if $B_1 \triangle B_2 = \{e, f\}$ and $\phi(e) \neq \phi(f)$, where $e \in B_1$ and $f \in B_2$. In this article, we characterize some conditions on ϕ and M, such that they guarantee the graph $G(\mathcal{B}(M), \phi)$ has a cycle or has no pendant vertex.

Keywords: basis graph, bicolor basis graph, pendant vertex.
Mathematics Subject Classification [2010]: Primary: 22D15, 43A10; Secondary: 43A20, 46H25

1 Introduction

Terminology will follow Oxley [4] and West [5]. Let M be a matroid. The set of bases of M is denoted by $\mathcal{B}(M)$. The basis graph of a matroid M, $G(\mathcal{B}(M))$, is a graph in which each vertex is labeled by a basis of M and two bases (vertices) are adjacent if one can be obtained from the other by a single element exchange. Let G be a connected graph. The tree graph of G is defined the basis graph of $M(G)$. In other word, two spanning trees T_1 and T_2 of G are adjacent in $G(\mathcal{B}(M(G)))$ if $T_1 - a + b = T_2$, where $a \in T_1$ and $b \in T_2$. Holzmann and Harary proved that $G(\mathcal{B}(M))$ is hamiltonian and therefore connected [3].

Figueroa defined a spanning subgraph of $G(\mathcal{B}(M))$ as following [2]; let $\phi : E(M) \longrightarrow \{1, 2\}$ be a surjection function. $G(\mathcal{B}(M), \phi)$, called a bicolor basis graph of M, is a spanning subgraph of $G(\mathcal{B}(M))$ such that two bases B_1 and B_2 of M are adjacent if $\phi(e) \neq \phi(f)$, where $B_1 \triangle B_2 = \{e, f\}$. The function ϕ is also called 2-coloring function.

They proved $G(\mathcal{B}(M), \phi)$ is connected if and only if M is a connected matroid. In particular they proved the following result.

Theorem 1.1. *Let M be a matroid with at least one circuit. Then, bicolor basis graph $G(\mathcal{B}(M), \phi)$ of M is connected for every 2-coloring ϕ of M, if and only if M is connected matroid.*

Let a be an element of M. If $\phi(a) \neq \phi(b)$ for every $b \in E(M) - a$, then we denote $G(\mathcal{B}(M), \phi)$ by $G(\mathcal{B}_a(M))$. As an example, see Figure 1.

Although connectivity of $G(\mathcal{B}_a(M))$ will be achieved by theorem 1.1, Chun et al proved it separately with a different method [1]. In fact, they defined two bases B_1 and B_2 are adjacent in $G(\mathcal{B}_a(M))$, if $a \in \{e, f\}$ where $B_1 \triangle B_2 = \{e, f\}$. This definition is equivalent with Figueroa's definition by the special ϕ as defined in the last paragraph.

It is easy to see that $G(\mathcal{B}(M), \phi)$ is a bipartite graph for every 2-coloring ϕ of M such that one part consists of the bases that have an odd number of a specific color and another part have an even number of this color. As $(E(M) - B_1) \triangle (E(M) - B_2) = B_1 \triangle B_2$, the following lemma easily results.

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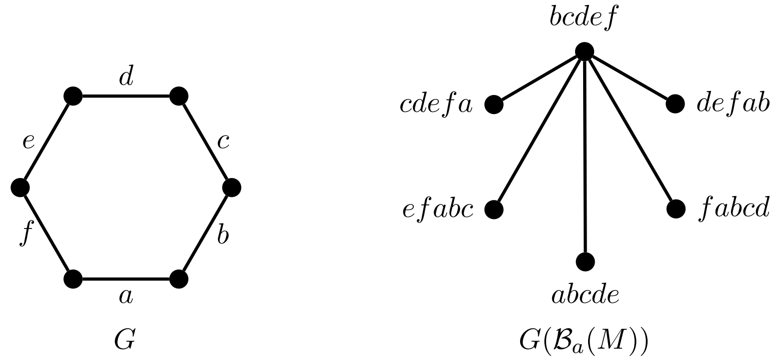

Figure 1: The graph G and The $G(\mathcal{B}_a(M))$.

Lemma 1.2. *Let M be a matroid and ϕ be a 2-coloring of M and $a \in E(M)$. Then*

i) $G(\mathcal{B}(M^*)) \cong G(\mathcal{B}(M))$

ii) $G(\mathcal{B}(M^*), \phi) \cong G(\mathcal{B}(M), \phi)$

iii) $G(\mathcal{B}_a(M^*)) \cong G(\mathcal{B}_a(M))$

We shall want to characterize some properties of $G(\mathcal{B}(M), \phi)$ related to various kind of ϕ and M. Many of our results are on $G(\mathcal{B}_a(M))$. We assume M is a connected matroid and G is a 2-connected simple graph.

2 Main Results

First, we shall want to prove two results related to the matroid and then we concentrate on the graph.

Lemma 2.1. *Let M be a matroid with at least 4 elements. Then there is a 2-coloring ϕ of M in which $G(\mathcal{B}(M), \phi)$ contains a cycle.*

Proof. First, suppose that each circuit and cocircuit of M has at most 3 elements. As M is connected and $r(M) + r^*(M) = |E(M)|$, then $|E(M)| = 4$. Since all of the connected matroids with 4 elements are recognized, it is easy to see that the theorem holds. By lemma 1.2, for every 2-coloring ϕ, $G(\mathcal{B}(M^*), \phi) \cong G(\mathcal{B}(M), \phi)$. Hence we may assume that M has a circuit with more than 3 elements. Suppose $C_n = \{e_1, e_2, \ldots, e_n\}$ is a circuit of M where $n \geq 4$. Clearly for all $e \in C_n$, $C_n - e$ is independent. Then there is an independent set I of M in which $I \cap C_n = \emptyset$ and $I \cup (C_n - e)$ is a basis of M. We define ϕ on M as the following way; $\phi(e_1) = \phi(e_3) = 1$, $\phi(e_2) = \phi(e_4) = 2$ and for the reminded elements of $E(M)$, ϕ is arbitrary. Consider the vertices $B_i = I \cup (C_n - e_i)$ in $G(\mathcal{B}(M), \phi)$ where $i \in \{1, 2, 3, 4\}$. These vertices construct a four elements cycle $B_1B_2B_3B_4B_1$ in $G(\mathcal{B}(M), \phi)$. □

In the next theorem, we use the two famous following propositions.

Proposition 2.2. *[4] If M is an n-connected matroid and $|E(M)| \geq 2(n-1)$, then all circuits and all cocircuits of M have at least n elements.*

Proposition 2.3. *[4] A 3-connected matroid M contains a set that is both a 3 elements circuit and a 3 elements cocircuit, if and only if $M \cong U_{2,4}$.*

We know that every 2-coloring function ϕ of M partitions $E(M)$ in two disjoint sets X and Y, where $\phi(X) = 1$ and $\phi(Y) = 2$. If $\min\{|X|, |Y|\} = 2$, then we use the notation φ instead of ϕ.

Theorem 2.4. *Let M be a 3-connected matroid and φ be a 2-coloring function of M. Then $G(\mathcal{B}(M), \varphi)$ has a cycle.*

Proof. The definition of φ ensures that $|E(M)| \geq 4$. Since the only 3-connected 4 elements matroid is $U_{2,4}$, one can easily check that the result holds in this case. Hence we may assume that $|E(M)| \geq 5$ and $M \ncong U_{2,4}$. Let a and b be the two same color elements having a different color than other elements. As M is 3-connected and therefore connected, by the proposition 2.2, M has a circuit and a cocircuit with at least three elements which contains $\{a, b\}$. We shall show that M has a circuit or a cocircuit with at least four elements containing $\{a, b\}$. Assume the contrary and suppose every circuit and cocircuit of M containing $\{a, b\}$ has three elements. Consider $C = \{a, b, c\}$ a 3 element circuit containing $\{a, b\}$. If C be a cocircuit too, then by proposition 2.3, $M \cong U_{2,4}$, a contradiction. Then M has a cocircuit $\{a, b, d\}$, where $d \neq c$. Thus $M \backslash d$ has $\{a, b\}$ as a cocircuit. Take $e \in E(M) - \{a, b, c, d\}$. Since $M \backslash d$ is connected, there is a circuit D containing $\{a, e\}$. Because $\{a, b\}$ is a cocircuit of $M \backslash d$ meeting D, we must have $\{a, b\} \subseteq D$. Hence $D = \{a, b, e\}$. Now $a \in C \cap D$ and therefore $\{b, c, e\} \subseteq (C \cup D) - \{a\}$ is a circuit of M meeting the cocircuit $\{a, b, d\}$ in a single element, a contradiction. Thus, by lemma 1.2 we may assume that, M has a four elements circuit containing $\{a, b\}$. Now a and b have the same color and at least two elements of this circuit have the another color. Therefore by the lemma 2.1, the result holds. □

Lemma 2.5. *Let M be a non-graphic uniform matroid of rank at least two. Then $G(\mathcal{B}(M), \phi)$ has no pendant vertex for every 2-coloring ϕ of M.*

Proof. Suppose $M = U_{m,n}$ and ϕ is a 2-coloring function of M. By the assumptions of the theorem we deduce $m \geq 2$ and $n - m \geq 2$. Let B be a arbitrary basis of M and e, f, g, h be elements of M such that $e, f \in B$ and $g, h \notin B$ and at least one of these four elements has a different color compared with the others. Notice that for every basis of M these conditions exist. Now B is adjacent to at least two vertices $(B - e) \cup g$, $(B - e) \cup h$, $(B - f) \cup g$ and $(B - f) \cup h$ in $G(\mathcal{B}(M), \phi)$. □

In the following results for notational convenience, we shall assume $M = M(G)$ where $M(G)$ is the cycle matroid of graph G.

Lemma 2.6. *Let G be a 2-connected simple graph and $a \in E(G)$. Then each vertex of $G(\mathcal{B}_a(M))$ that does not contain a has a degree at least 2.*

Proof. Consider $a \in E(G)$ and let T be a spanning tree of G such that $a \notin T$. $T + a$ contains a cycle that has more than 2 edges. Let e and f be two edges of this cycle in which $a \notin \{e, f\}$. Now $T_1 = T + a - e$ and $T_2 = T + a - f$ are two spanning trees of G such that $T \triangle T_1 = \{a, e\}$ and $T \triangle T_2 = \{a, f\}$. Thus T is adjacent to T_1 and T_2 in $G(\mathcal{B}_a(M))$. □

Theorem 2.7. *Let $n \geq 4$ and G be a Hamiltonian simple graph of order n with Hamiltonian cycle C_n. Let $G - L = C_n$ where L is non-empty subset of $E(G)$. Then if $a \in L$, $G(\mathcal{B}_a(M))$ has no pendant vertex.*

Proof. Let G be the graph as specified in the theorem. Suppose T is a spanning tree of G containing a. Evidently, $T - a$ is disconnected and it has two components with vertex set S and $\overline{S}$. Since G has C_n as a spanning subgraph, thus it is 2-connected and the vertex cut $[S, \overline{S}]$ contains two edges e, f of C_n. As $a \notin C_n$, so $a \notin \{e, f\}$. Therefore $T_1 = T - a + e$ and $T_2 = T - a + f$ are two spanning trees of G and $T \triangle T_1 = \{a, e\}$ and $T \triangle T_2 = \{a, f\}$. Hence the vertex T in $G(\mathcal{B}_a(M))$ is adjacent to vertices T_1 and T_2. Now suppose T does not contain a. by the lemma 2.6 we deduce that the theorem holds. □

Corollary 2.8. *Let $n \geq 4$ and G be a complete graph of order n. If $a \in E(K_n)$ then $G(\mathcal{B}_a(M))$ has no pendant vertex.*

Proof. For all $a \in E(K_n)$, $K_n - a$ is a Hamiltonian graph. Therefore by the theorem 2.7, $G(\mathcal{B}_a(M))$ has no pendant vertex. □

Theorem 2.9. *Let G be a 2-connected simple graph. Then G has a 2-degree vertex if and only if for some $a \in E(G)$, $G(\mathcal{B}_a(M))$ has a pendant vertex.*

Proof. Let a and b be two incident edges of a 2-degree vertex v. Let T be a spanning tree of G such that $a \in T$ and $b \notin T$. For $e \in E(G) - T - b$, $T + e$ contains a cycle, C. Since v has degree two, every cycle that contains a must contain b. Thus $a \notin C$. Then $T + e - a$ contains C and it is not a spanning tree of G. Therefore T is just adjacent to spanning tree $T + b - a$ in $G(\mathcal{B}_a(M))$.

Conversely, suppose $G(\mathcal{B}_a(M))$ has a pendant vertex T for some $a \in E(G)$. By the lemma 2.6, $a \in T$. Let T_1 be the only adjacent vertex of T. Hence $T \triangle T_1 = \{a, b\}$ where b is an edge of T_1. As T is a one degree vertex so for all $e \in E(G) - T - b$, $T - a + e$ is not a spanning tree of G and then contains a cycle. Therefore this cycle contained in $T + e$ and a is not belong to it. Since e is an arbitrary element and $G - b = T \cup \{e_i\}$ where $e_i \in G - T - b$. we conclude that $G - b$ has no cycle contains a. Then every cycle of G that has b must have a too. Therefore at least one endpoint of a must be a two degree vertex. □

Corollary 2.10. *Let G be a 2-connected simple graph and $\delta(G) \geq 3$. Then for all $a \in E(G)$, $G(\mathcal{B}_a(M))$ has no pendant vertex.*

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On subgroup inclusion graph of free groups with finite rank

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Abstract

In this paper, we continue the study of the subgroup inclusion graph $In(F)$ on a free group with finite rank F, where the vertex set is the collection of non-trivial proper subgroups of a free group with finite rank and two vertices are adjacent if one is contained in other. We show that $In(F)$ is perfect.

Keywords: Graph, Subgroup, Free group.
Mathematics Subject Classification [2010]: Primary: 05C25; Secondary: 05C69

1 Introduction

Associating graphs to algebraic structures for characterizing the algebraic structures with graphs and vice versa is an interesting area in graph theory. To date, a lot of research, e.g. [1, 2, 5] has been done in connecting graph structures to various algebraic objects. By a graph $G(V;E)$, we mean a non-empty set V and a symmetric binary relation (possibly empty) E on V. The set V is called the set of vertices and E is called the set of edges of G. Two elements $u, v \in V$ are said to be adjacent if $(u, v) \in E$. $H = (W, F)$ is called a subgraph of G if H itself is a graph and $\emptyset \neq W \subseteq V$ and $F \subseteq E$. If V is finite, the graph G is said to be finite, otherwise it is infinite. Complement or inverse of a graph G is a graph G^c on the same vertices such that two distinct vertices of G^c are adjacent if and only if they are not adjacent in G. If all the vertices of G are pairwise adjacent, then G is said to be complete. A complete subgraph of a graph G is called a clique. A maximal clique is a clique which is maximal with respect to inclusion. The clique number of G, written as $\omega(G)$, is the maximum size of a clique in G. The chromatic number of G, denoted as $\chi(G)$, is the minimum number of colours needed to label the vertices so that the adjacent vertices receive different colours. It is known that for any graph G, $\chi(G) \geq \omega(G)$. A graph G is called perfect if for any induced subgraph H of G, $\chi(G) = \omega(G)$. In this paper, we continue the study of the subgroup inclusion graph $In(F)$ of a free group with finite rank F, introduced in [4].

2 Main Results

Let F be a free abelian group with finite rank of rank greater than 1. The author define a graph $In(F) = (V;E)$ as follows: $V =$ the collection of nontrivial proper subgroups of F and for $W_1, W_2 \in V$, $W_1 \sim W_2$ or $(W_1, W_2) \in E$ if either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. Since, $rank(F) > 1$, $V \neq \emptyset$.

Lemma 2.1. *If F is a free abelian Group of finite rank and W is a subgroup of F with dimension greater than 1, then $In(W)$ is a subgraph of $In(F)$.*

Proof. It follows from the definition of $In(F)$ and the fact that every subgroup of W is a subgroup of F. □

[1]speaker

Lemma 2.2. *Let M be a clique in $In(F)$, then M is a chain of nontrivial proper subgroups of F.*

Proof. Suppose that $W_1, W_2 \in M$. Since M is a clique, either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. Thus, any two elements in M are comparable, as required. □

Theorem 2.3. *$rank(F) = n$ if and only if $\omega(In(F)) = n - 1$.*

Proof. Follows from [4]. □

Theorem 2.4. *If $rank(F) = n$, then $\chi(In(F)) = n - 1$.*

Proof. Suppose that $rank(F) = n$. Hence clique number of $In(F)$ is $n - 1$, and so $\chi(In(F)) \geq n - 1$. Assume that $W \in V$ and color W with the $i-$th color if $rank(W) = i$. This is a proper coloring, as Lemma 2.1 ensures that two $i-$dimensional subgroups are never, as required. □

It is shown that $In(F)$ is weakly perfect, i.e. $\omega(In(F)) = \chi(In(F)) = n - 1$. In this following, we show that $In(F)$ is perfect, i.e. $\omega(H) = \chi(H)$ for every induced subgraph H of $In(F)$. In [3] Chudnovsky, et.al. were shown that, "a graph G is perfect if and only if neither G nor it's complement contains an odd cycle of length at least 5 as an induced subgraph".

Lemma 2.5. *$In(F)$ has no induced cycle of odd length greater than* 3.

Proof. Let $C : W_1 \sim W_2 \sim W_3 \sim W_{2k+1} \sim W_1$ be an induced cycle in $In(F)$ with $k \geq 2$. Since $W_1 \sim W_2$, without loss of generality, let $W_1 \subseteq W_2$. Since $W_2 \sim W_3$, $W_2 \subseteq W_3$ or $W_3 \subseteq W_2$. If $W_2 \subseteq W_3$, then $W_1 \subseteq W_3$, i.e. $W_1 \sim W_3$, a contradiction to the fact that C is an induced cycle in $In(V)$. Thus $W_3 \subseteq W_2$. Proceeding in a similar manner we get

$$W_1 \subseteq W_2 \supseteq W_3 \subseteq W_5 \cdots \supseteq W_{2k+1} \subseteq W_1.$$

Therefore $W_{2k+1} \subseteq W_1 \subseteq W_2$, ansd so $W_2 \sim W_{2k+1}$ is a chord, a contradiction. □

Lemma 2.6. *The complement of $In(F)$, i.e. $(In(F))^c$ has no induced cycle of odd length greater than* 3.

Proof. Let $C : W_1 \sim W_2 \sim W_3 \sim W_{2k+1} \sim W_1$ be an induced cycle in $(In(F))^c$ with $k \geq 2$. For sake of definiteness and for later use, let us call W_{2k+1}, the final vertex in C. Since $W_1 \sim W_2$, $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$. Since $W_1 \nsim W_3$, without loss of generality, let $W_1 \subseteq W_3$. Since W_1 is also not adjacent to W_4, either $W_1 \subseteq W_4$ or $W_4 \subseteq W_1$. If $W_4 \subseteq W_1$, then $W_4 \subseteq W_3$ and hence $W_3 \nsim W_4$, a contradiction. Thus $W_1 \subseteq W_4$. Since $W_2 \nsim W_4$, either $W_2 \subseteq W_4$ or $W_4 \subseteq W_2$. If $W_4 \subseteq W_2$, then $W_1 \subseteq W_2$, and so $W_1 \nsim W_2$, a contradiction. Thus $W_2 \subseteq W_4$. Since $W_2 \nsim W_5$, $W_2 \subseteq W_5$. Finally, since $W_3 \nsim W_5$, either $W_3 \subseteq W_5$ or $W_5 \subseteq W_3$. Hence $W_1 \subseteq W_5$, and so $W_1 \nsim W_5$, i.e. W_5 is not the final vertex in C. Thus, there exist at least two more vertices W_6 and W_7 in the induced cycle C. However, continuing in the same way, we can show that $W_1 \subseteq W_7$, and so $W_1 \nsim W_7$, i.e. W_7 is not the final vertex in C. Arguing similarly, the induced odd cycle C has two more vertices W_8 and W_9. Observe that this process continues indefinitely and as a result we cannot get an induced odd cycle of finite length in $(In(F))^c$, a contradiction to the finiteness of length of C. Thus $(In(F))^c$ has no induced cycle of odd length greater than 3. □

Corollary 2.7. *$In(F)$ is perfect.*

Proof. Follows from [3] and Lemmas 2.5 and 2.6. □

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Some graphical invariants related to binomial edge ideals

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Abstract

In this talk we aim to build some unexpected links between some homological invariants of binomial edge ideals, and some graphical invariants. In this direction, we introduce the concept of compatible maps, which are defined from the set of all graphs to the set of non-negative integers that admit certain properties. Then, this concept will be exploited to establish some combinatorial bounds for some of the homological invariants of binomial edge ideals.

Keywords: Binomial edge ideals, compatible maps, graphical invariants, homological invariants.

Mathematics Subject Classification [2010]: Primary: 05E40; Secondary: 13D02.

1 Introduction

Over the last two decades, the study of ideals with combinatorial origins has been an appealing trend in commutative algebra. One of the most well-studied types of such ideals which has attracted special attention in the literature is the binomial edge ideal of a graph.

Let G be a graph on $[n] = \{1, \dots, n\}$ and $S = \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_n]$ be the polynomial ring over a field $\mathbb{K}$. Then, the *binomial edge ideal* associated to G, denoted by J_G, is the ideal in S generated by all the quadratic binomials of the form $f_{ij} = x_i y_j - x_j y_i$, where $\{i, j\} \in E(G)$ and $1 \leq i < j \leq n$. This class of ideals was introduced in 2010 by Herzog, Hibi, Hreinsdóttir, Kahle and Rauh in [1], and independently by Ohtani in [2], as a natural generalization of determinantal ideals, as well as ideals generated by adjacent 2-minors of a $2 \times n$-matrix of indeterminates. Since then, many of algebraic and homological properties and invariants of J_G were investigated by several authors. The reader could see the survey article [6] for some of the efforts in this direction.

[1] speaker

2 Main Results

Two of the most important homological invariants associated to binomial edge ideals are the *depth*, and the *Castelnuovo-Mumford regularity*.

In this talk, after reviewing some of the results in the literature, we first establish a combinatorial lower bound for the depth of S/J_G. To do so, we introduce the concept of *d-compatible maps*, and define a graphical invariant related to this concept. Let G be a graph on $[n]$ and $v \in [n]$. Associated to the vertex v, there is a graph, denoted by G_v, with the vertex set $V(G)$ and the edge set

$$E(G) \cup \{\{u, w\} : \{u, w\} \subseteq N_G(v)\}.$$

Note that by the definition, it is clear that v is a free (or simplicial) vertex of the graph G_v.

Now we are ready to define the notion of a d-compatible map as follows:

Definition 2.1. *Let $\mathcal{G}$ be the set of all graphs. A map $\psi : \mathcal{G} \longrightarrow \mathbb{N}_0$ is called* d-compatible*, if it satisfies the following conditions:*

(a) if $G = \dot{\bigcup}_{i=1}^{t} K_{n_i}$, where $n_i \geq 1$ for every $1 \leq i \leq t$, then $\psi(G) \leq t + \sum_{i=1}^{t} n_i$;

(b) if $G \neq \dot{\bigcup}_{i=1}^{t} K_{n_i}$, then there exists a non-free vertex $v \in V(G)$ such that

(1) $\psi(G - v) \geq \psi(G)$, and
(2) $\psi(G_v) \geq \psi(G)$, and
(3) $\psi(G_v - v) \geq \psi(G) - 1$.

The following theorem provides a general lower bound for the depth of binomial edge ideals.

Theorem 2.2. *Let G be a graph on $[n]$ and ψ be a d-compatible map. Then*

$$\operatorname{depth} S/J_G \geq \psi(G).$$

Now, we are going to provide a combinatorial d-compatible map. Let G be a graph on $[n]$ with the connected components $G_1, \ldots, G_t$. Then we set

$$d(G) := \mathfrak{i}(G) + \sum_{i=1}^{t} \operatorname{diam}(G_i),$$

where $\mathfrak{i}(G)$ denotes the number of isolated vertices of G, and $\operatorname{diam}(G_i)$ denotes the *diameter* of the graph G_i for every i. Now, in the next theorem, we provide a d-compatible map given by $\xi(G)$. Here, $f(G)$ denotes the number of free vertices of G.

Theorem 2.3. *The map $\xi : \mathcal{G} \longrightarrow \mathbb{N}_0$ defined by*

$$\xi(G) = f(G) + d(G), \text{ for every } G \in \mathcal{G}$$

is d-compatible.

Now, combining Theorem 2.2 and Theorem 2.3 we get the following combinatorial lower bound for the depth of binomial edge ideals.

Theorem 2.4. *Let G be a graph on $[n]$. Then*

$$\operatorname{depth} S/J_G \geq \xi(G).$$

In particular, if G is connected, then

$$\operatorname{depth} S/J_G \geq f(G) + \operatorname{diam}(G).$$

Finally, if time permits, we will discuss about the other interesting homological invariants of binomial edge ideals, which is the Castelnuovo-Mumford regularity. In particular, we may introduce the concept of r-compatible maps, and also a new graphical invariant to tackle a conjecture in this direction.

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The enhanced power graph and the commuting graph

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Abstract

The purpose of this note is to define a graph whose vertex set is a finite group G, whose edge set is contained in that of the commuting graph of G and contains the enhanced power graph of G. We call this graph the deep commuting graph of G. Two elements of G are joined in the deep commuting graph if and only if their inverse images in every central extension of G commute.

We give conditions for the graph to be equal to either of the enhanced power graph and the commuting graph, and show that the automorphism group of G acts as automorphisms of the deep commuting graph.

Keywords: power graph, commuting graph
Mathematics Subject Classification [2010]: 05C10, 05C30

1 Introduction

Let G be a finite group. Among a number of graphs defined on the vertex set G reflecting some algebraic properties of the group, two which have been studied are the following:

- the *commuting graph*, $\mathrm{Com}(G)$, first defined by Brauer and Fowler [2], in which two elements x and y are joined if they commute;
- the *enhanced power graph*, $\mathrm{EPow}(G)$, first introduced by Aalipour *et al.* [1], in which x and y are joined if they generate a cyclic subgroup of G.

It is clear that the edge set of the enhanced power graph is contained in that of the commuting graph. (That is, $\mathrm{EPow}(G)$ is a *spanning subgraph* of $\mathrm{Com}(G)$.) The purpose of this note is to define a graph whose edge set is between these two (contained in the first and containing the second). We will call it the *deep commuting graph*, for reasons which will hopefully become clear, and denote it by $\mathrm{DCom}(G)$.

The definition requires some preliminary discussion of Schur covers and isoclinism.

2 Covers and isoclinism

In a group G, the *center* $Z(G)$ is the subgroup consisting of elements which commute with every element of G. The *derived group* G' is the subgroup generated by all *commutators* $[x,y] = x^{-1}y^{-1}xy$ for $x, y \in G$.

A *central extension* of a finite group G is a finite group H with a subgroup $Z \leqslant Z(H)$ such that $H/Z \cong G$. We will think of G as a quotient of H, in other words, the image of a homomorphism whose kernel is Z. If $Z \leqslant Z(H) \cap H'$, it is called a *stem extension*. The subgroup Z is the *kernel* of the extension.

The deep commuting graph can be defined to have vertex set G, with an edge joining x and y if and only if $x \neq y$ and the preimages of x and y commute in every central extension of G. (It is sometimes more

[1]speaker

natural to allow loops, but it makes no difference here.) However, we will give a more explicit definition shortly.

As a temporary notation, we define the *relative commuting graph* of G by Z as the graph with vertex set G, in which x and y are joined if their inverse images in H commute.

Schur showed that every finite group has a unique *Schur multiplier* $M(G)$, the central subgroup in the largest possible stem extension of G. There are several other characterisations of $M(G)$: for example, it is the first homology group $H_1(G, \mathbb{Z})$, or the first cohomology group $H^1(G, \mathbb{C}^\times)$, or the quotient $(R \cap F')/[R, F]$ where G has a presentation F/R as quotient of a free group. The corresponding stem extension H is called a *Schur cover* of G; it is not always uniquely defined by G. For example, the dihedral and quaternion groups of order 8 are both Schur covers of the Klein group of order 4.

However, Jones and Wiegold [3] showed that a relation weaker than uniqueness holds, namely that of isoclinism. This means that the commutation maps γ from $H/Z(H) \times H/Z(H)$ to H' defined by $\gamma(Z(H)x, Z(H)y) = [x, y]$ are essentially the same in the two groups. (Note that this is independent of the choice of coset representatives.) More formally, two groups H_1 and H_2 are said to be *isoclinic* if there exist isomorphisms $\phi\colon H_1/Z(H_1) \to H_2/Z(H_2)$ and $\psi\colon H_1' \to H_2'$ such that, for all $x, y \in H_1$,

$$\psi(\gamma_1(Z(H_1)x, Z(H_1)y)) = \gamma_2(\phi(Z(H_1)x), \phi(Z(H_1)y)), \tag{1}$$

where γ_1 and γ_2 are the commutation maps associated with H_1 and H_2; in other words, the isomorphisms respect the commutation map. The pair (ϕ, ψ) is an *isoclinism* from H_1 to H_2.

Example 2.1. As noted, the Klein group of order 4 has two Schur covers up to isomorphism, the dihedral and quaternion groups of order 8. In fact there are three different covers isomorphic to the dihedral group, since any one of the three involutions in the Klein group may be the one that lifts to an element of order 4 in the cover, but a unique cover isomorphic to the quaternion group. However, in all cases, the commuting graph of the cover is the lexicographic product of a complete graph of order 2 with a star $K_{1,3}$; so the deep commuting graph of the Klein group is isomorphic to $K_{1,3}$ (which is in fact equal to the enhanced power graph of this group).

3 Commuting graph and commuting probability

As a preliminary to this section, we note a connection between the commuting graph and the commuting probability of a group G.

Let G be a finite group. Then G acts on itself by conjugation; the stabiliser of a point x is its centraliser (the set of neighbours of x in the commuting graph), while the orbits are conjugacy classes. So the Orbit-counting Lemma shows that the number of conjugacy classes is equal to the average valency of the commuting graph. Dividing through by $|G|$, we see that the proportion of all ordered pairs which commute is equal to the ratio of the number of conjugacy classes to $|G|$. This fraction is called the *commuting probability* of G. We denote it by $\kappa(G)$, and note that it is the ordered edge-density of the commuting graph (with a loop at each vertex).

Proposition 3.1. *Suppose that $\pi_i : H_i \to G$ are epimorphisms corresponding to central extensions of G for $i = 1, 2$. Suppose that there exists an epimorphism $\phi : H_1 \to H_2$ such that $\pi_2 \circ \phi = \pi_1$. Then $\kappa(H_1) \leqslant \kappa(H_2)$.*

Proof. This follows from the fact that, for $x, y \in H_1$, if x and y commute then $\phi(x)$ and $\phi(y)$ commute in H_2. □ □

Though it is not necessary for our argument, we note that in the cited paper Eberhard proved that values of the commuting probability for finite groups are well-ordered by the reverse of the usual order on the unit interval. Thus, in the situation of the above proposition, either $\kappa(H_1) = \kappa(H_2)$, or $\kappa(H_1) \leqslant \kappa(H_2) - \epsilon$, where ϵ depends only on $\kappa(H_2)$.

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Cubic edge-transitive graphs of order $12p^3$

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Abstract

A simple undirected graph is said to be semisymmetric if it is regular and edge-transitive but not vertex-transitive. Let p be a prime. Then, by Malnic et al. [18, 21 and 22], there exists a unique cubic semisymmetric cubic graph of order $2p^3$ and by Feng et al. [3] a cubic semisymmetric graph of order $6p^3$ exists if and only if $p-1$ is divisible by 3. In this paper, we prove that every cubic edge-transitive graph of order $12p^3$ is vertex-transitive and so there is no cubic semisymmetric graph of order $12p^3$.

Keywords: Middle neighborhood graph, Domination number, Independent domination.
Mathematics Subject Classification [2010]: 05C25, 20B25.

1 Introduction

Symmetric and semisymmetric graphs are used in many scientific domains, especially parallel computation and interconnection networks. The industry and the research world make a huge usage of such graphs. Throughout this paper, graphs are assumed to be finite, simple, undirected and connected. For the group-theoretic concepts and notations not defined here we refer the reader to [3, 5, 13].

The study of semisymmetric graphs was initiated by Folkman [8]. He constructed several infinite families of such graphs and posed eight open problems. By now, the answers to most of Folkman's open problems are known. By now, cubic semisymmetric graphs of orders $2p$, $2p^2$, $4p$, $4p^2$, $6p$, $6p^2$, $8p$, $8p^2$, $10p$, $10p^2$, $12p$, $12p^2$, $14p$, $14p^2$, $2p^3$, $4p^3$, $6p^3$, $8p^3$, $10p^3$ for each prime p were have been investigated (See [8, 10, 1, 2, 11, 7, 9]). The following theorem is the main result of this paper which states there is no semisymmetric cubic graph of order $12p^3$.

Theorem 1.1. *Let p be a prime. Then every edge-transitive cubic graph of order $12p^3$ is vertex-transitive.*

2 Preliminaries

Let X be a graph and N a subgroup of $\mathrm{Aut}(X)$. Denote by X_N *the quotient graph* corresponding to the orbits of N, that is the graph having the orbits of N as vertices with two adjacent orbits in X_N whenever there is an edge between those orbits in X. The following proposition is the special case of [10, $Lemma 3.2$] .

Proposition 2.1. *Let X be a connected cubic G-semisymmetric graph with bipartition sets $U(X)$ and $W(X)$. Moreover suppose that N is a normal subgroup of G. If N is intransitive on bipartition sets, then N acts semiregularly on both $U(X)$ and $W(X)$, and X is a regular covering of a G/N-semisymmetric graph with the covering transformation group N.*

[1]speaker

Proposition 2.2. [*11, Proposition*2.4] *The vertex stabilizers of a connected G-edge-transitive cubic graph X have order 2^r. 3, where $r \geq 0$. Moreover if u and v are two adjacent vertices, then $|G : \langle G_u, G_v\rangle| \leq 2$, and the edge stabilizer $G_u \bigcap G_v$ is a common Sylow 2-subgroup of G_u and G_v.*

By Malnič et al. [11, *Theorem*1.1], every cubic edge-transitive graph of order $2p^3$ is vertex-transitive for a prime p greater than 3, and by [6, *Theorem*3.2], every cubic symmetric graph of order $2p^3$ is a normal Cayley graph on a group of order p^3 for an odd prime p. Thus, we have the following proposition.

Proposition 2.3. *Let $p \geq 5$ be a prime and X a cubic edge-transitive graph of order $2p^3$. Then $Aut(X)$ has a normal Sylow p-subgroup.*

Proposition 2.4. [*9, Proposition*2.5] *Let G be a three-prime-factor simple group. Then G is one of the following groups :*

$$A_5,\ A_6,\ PSL_2(7),\ PSL_2(8),\ PSL_2(17),\ PSL_3(3),\ PSU_3(3),\ PSU_4(2) \qquad (1)$$

3 Main result

By [3], for $p = 2$ or 3, there is no connected cubic semisymmetric graph of order $12p^3$. Thus we may assume that $p \geq 5$.

Lemma 3.1. *Let $p \geq 5$ be a prime and let X be a connected cubic semisymmetric graph of order $12p^3$. Then Aut(X) has a normal Sylow p-subgroup.*

Proof. Let X be a cubic graph satisfying the assumptions and $A :=$Aut(X). Since semisymmetric graphs are bipartite, one may take $L(X)$ and $R(X)$ the bipartite sets of X. Clearly $|L(X)| = |R(X)| = 6p^3$. By Proposition 2.2, $|A| = 2^r.\ 3^2.\ p^3$ for some integer $r \geq 1$. Let $O_p(A)$ be a maximal normal p-subgroup of A. We prove that $|O_p(A)| = p^3$. Let N be a minimal normal subgroup of A. Assume that N is unsolvable. Then $N = T \times T \times ... \times T = T^k$, where T is a non-abelian simple group. Since A is a $\{2,\ 3,\ p\}$-group, T is one of the groups listed in (1) with orders $2^2.\ 3.\ 5,\ 2^3.\ 3^2.\ 5,\ 2^4.\ 3.\ 7,\ 2^4.\ 3^2.\ 7,\ 2^4.\ 3^2.\ 17,\ 2^4.\ 3^3.\ 13,\ 2^5.\ 3^3.\ 7$ and $2^6.\ 3^4.\ 7$, respectively. Since 3^3 does not divide $|A|$, one has $k \leq 2$ and hence $p^3 \nmid |N|$. It follows that N is intransitive on $L(X)$ and $R(X)$. By Proposition 2.1, N is semiregular on $L(X)$ (and $R(X)$), which implies that $|N| \mid 6p^3$, a contradiction. Thus N is solvable and hence elementary abelian. Again by Proposition 2.1, N is semiregular on $L(X)$ (and $R(X)$). Moreover, the quotient graph X_N of X corresponding to the orbits of N is A/N-semisymmetric. The semiregularity of N implies that $|N| \mid 6p^3$.

Suppose first that $O_p(A) = 1$. Since $O_p(A) = 1$ and $|N| \mid 6p^3$ one has two cases $N \cong Z_2$ or Z_3. The details of the proof of two cases are similar and hence we investigate only the case $N \cong Z_2$ and omit the proof of the other case.

Let $N \cong Z_2$. Then X_N is a cubic graph of order $6p^3$ with A/N as a semisymmetric subgroup of Aut(X_N). We denote by $L(X_N)$ and $R(X_N)$ the bipartite sets of X_N. Let L/N be a minimal normal subgroup of A/N. By the same argument as in the preceding paragraph we may prove that L/N is solvable and hence elementary abelian. By Proposition 2.1, L/N is semiregular on $L(X_N)$ and $R(X_N)$ and hence $|L/N| = 3$, because $O_p(A) = 1$. Thus $|T| = 6$. By Proposition 2.1, A/L is a semisymmetric subgroup of Aut(X_L). Note that $|A/L| = 2^{r-1}.\ 3.\ p^3$ and $|V(X_L)| = 2p^3$. Let P be a Sylow p-subgroup of A. By Proposition 2.3, Aut(X_L) has a normal Sylow p-subgroup and hence $PL/L \lhd A/L$. Consequently, $PL \lhd A$. It is easy to see that in this case P is characteristic in PL, forcing that $P \trianglelefteq A$, which is a contradiction.

Now suppose that $|O_p(A)| = p^s$ for $s = 1$ or 2. Let $G = A/O_p(A)$ and $Y = X_{O_p(A)}$. Then $|G| = 2^r.\ 3^2.\ p^{3-s}$. By Proposition 2.1, Y is G-semisymmetric, and the two partite sets $L(Y)$ and $R(Y)$ of Y have cardinality $6p^{3-s}$. Let H be a minimal normal subgroup of G. If H is solvable then by a similar argument as above we can get a contradiction (replacing A and N by G and H, respectively).

Suppose that H is unsolvable. Then $H = T^k$ for a non-abelian simple group H. Consider the two cases $|G| = 2^r.\ 3^2.\ p^2$ or $|G| = 2^r.\ 3^2.\ p$, respectively. Clearly, $k \leq 2$. Let $|G| = 2^r.\ 3^2.\ p^2$. Then $|L(Y)| = |R(Y)| = 6p^2$. If $k = 1$ then H is a non-abelian simple group listed in (1). Thus $p^2 \nmid |H|$ and by

Proposition 2.1, H is semiregular on $L(Y)$ (and $R(Y)$), implying $|H| \mid 6p^2$, a contradiction. Hence $k = 2$ and Y is H-semisymmetric. This leads to a contradiction as the preceding case (replacing G and H by H and N, respectively). Let $|G| = 2^r.\ 3^2.\ p$. Then $|L(Y)| = |R(Y)| = 6p$. Since H cannot act semiregularly on $L(Y)$ and $R(Y)$ for any $v \in V(Y)$, $|H_v| \neq 1$. Note that G_v acts primitively on the neighborhood of v in Y . Since $H_v \trianglelefteq G_v$, $3 \mid |H_v|$, implying $3^2||H|$. Since $3^3 \nmid |G|$, by the simple groups listed in (1), $H \cong A_6, PSL_2(8)$ or $PSL_2(17)$, which means that $|H_v| = 12$ or 24 and hence Y is at least 3-regular. Then graph Y has order $60, 84$ or 204. By [4], there is no connected cubic semisymmetric graph of order $60, 84$ or 204. Hence Y is symmetric. By [3], a cubic symmetric graph of order 60 or 84 is 2-regular. Thus Y has order 204. Again by [7], Y is 4-regular and so Aut(Y)= H, which is impossible because Aut(Y) is transitive on vertices of Y , but H is not. □

Proof of Theorem 1.1 Suppose that X is not vertex-transitive. Let A =Aut(X) and let P be a Sylow p-subgroup of A. By Lemma 3.1, P is a normal subgroup of A. Then by Proposition 2.1, X is a P-covering of a graph with 12 vertex. Moreover A/P is semisymmetric on this graph. This contradicts the fact that every cubic graph of order 12 is not edge-transitive. Therefore X is vertex-transitive and the proof is complete.■

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Enumeration of accurate dominating sets

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Abstract

Let $G = (V, E)$ be a simple graph. A dominating set of G is a subset $D \subseteq V$ such that every vertex not in D is adjacent to at least one vertex in D. The cardinality of a smallest dominating set of G, denoted by $\gamma(G)$, is the domination number of G. A dominating set D is a accurate dominating set of G, if no $|D|$-element subset of $V \setminus D$ is a dominating set of G. The accurate domination number, $\gamma_a(G)$, is the cardinality of a smallest accurate dominating set D. In this paper, after presenting preliminaries, we count the number of accurate dominating sets of some specific graphs.

Keywords: domination number, accurate dominating set, path, cycle.
Mathematics Subject Classification [2010]: Primary: 05C69, 05C05; Secondary: 05C75

1 Introduction

Let $G = (V, E)$ be a simple graph with n vertices. Throughout this paper we consider only simple graphs. A set $S \subseteq V(G)$ is a dominating set if every vertex in $V(G)\setminus S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. There are various domination numbers in the literature. For a detailed treatment of domination theory, the reader is referred to [6].

An accurate dominating set of G is a dominating set D of G such that no $|D|$-element subset of $V \setminus D$ is a dominating set of G. The accurate domination number of G, $\gamma_a(G)$, is the cardinality of a smallest accurate dominating set of G. A dominating set with cardinality $\gamma(G)$ is called a *γ-set*. Also an accurate dominating set of G of cardinality $\gamma_a(G)$ is called a γ_a-set of G. The accurate domination in graphs was introduced by Kulli and Kattimani [8], and further studied in a number of papers (see, for example, [4, 5]).

The concept of domination and related invariants have been generalized in many ways. Among the best know generalizations are total, independent, and connected dominating, each of them with the corresponding domination number. Most of the papers published so far deal with structural aspects of domination, trying to determine exact expressions for $\gamma(G)$ or some upper and/or lower bounds for it. There were no paper concerned with the enumerative side of the problem by 2008.

Regarding to enumerative side of dominating sets, Saeid Alikhani and Yee-hock Peng [3], have introduced the domination polynomial of a graph. The domination polynomial of graph G is the generating function for the number of dominating sets of G, i.e., $D(G, x) = \sum_{i=1}^{|V(G)|} d(G, i)x^i$ (see [1, 3]). This polynomial and its roots has been actively studied in recent years (see for example [7, 9]). It is natural to count the number of another kind of dominating sets ([2]). Let $\mathcal{D}_a(G, i)$ be the family of accurate dominating sets of a graph G with cardinality i and let $d_a(G, i) = |\mathcal{D}_a(G, i)|$. The generating function for the number of accurate dominating sets of G is denoted by $D_a(G, x)$.

[1]speaker

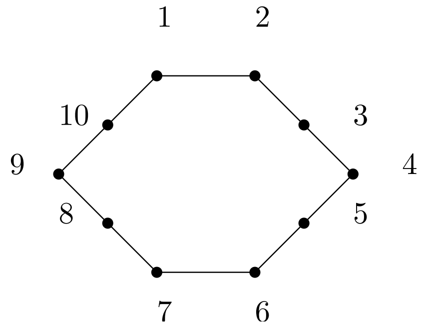

Figure 1: The graph C_{10} with $V(C_{10}) = \{1, 2, ..., 10\}$.

The corona of two graphs G_1 and G_2, is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added. The join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv|u \in V(G_1) \text{ and } v \in V(G_2)\}$.

In this paper, we study the number of accurate dominating sets of specific graphs.

2 Main Results

By the definition of accurate dominating set, every accurate dominating set is a dominating set (and so $\gamma(G) \leq \gamma_a(G)$) but the converse is not true. In other words, in some graphs there exists dominating sets which are not accurate dominating sets. For example in the cycle C_{10} with $V(C_{10}) = \{1, 2, ..., 10\}$ (see Figure 1), the set $D = \{1, 2, 3, 5, 8\}$ is a dominating set of C_{10} which is not accurate dominating set.

In this section, we study the number of accurate dominating sets of specific graphs. First we state some known results.

Lemma 2.1. [4]

(i) For every natural number n, $\gamma_a(K_n) = \lfloor \frac{n}{2} \rfloor + 1$.

(ii) For every natural number n, $\gamma_a(K_{n,n}) = n + 1$.

(iii) For $n > m \geq 1$, $\gamma_a(K_{m,n}) = m$.

(iv) For $n \geq 3$, $\gamma_a(C_n) = \lfloor \frac{n}{3} \rfloor - \lfloor \frac{3}{n} \rfloor + 2$.

(v) $\gamma_a(P_n) = \lceil \frac{n}{3} \rceil$ unless $n \in \{2, 4\}$.

Theorem 2.2. *Let G be a graph of order n.*

(i) If D is a dominating set with $|D| \geq \lfloor \frac{n}{2} \rfloor + 1$, then D is an accurate dominating set of G.

(ii) $d(G, i) = d_a(G, i)$ for every $i \geq \lfloor \frac{n}{2} \rfloor + 1$.

(iii) If $\gamma_a(G) > \lfloor \frac{n}{2} \rfloor$, then $D(G, x) = D_a(G, x)$.

By Theorem 2.2, for a graph G of order n, we shall find $d_a(G, i)$ for $\gamma_a(G) \leq i \leq \lfloor \frac{n}{2} \rfloor$. Because for $i > \lfloor \frac{n}{2} \rfloor$, we can use results on the domination polynomial to obtain $d_a(G, i)$. Note that there is a graph G of order n, such that G has accurate dominating sets D with $|D| \leq \lfloor \frac{n}{2} \rfloor$. For example we state the following theorem:

Theorem 2.3. *Let F_n be the friendship graph which is the join of K_1 and nK_2.*

(i) $\gamma_a(F_n) = 1$.

(ii) For every $i \in \mathbb{N}$, $d_a(F_n, i) = \binom{n}{i-n} 2^{2n-i} + \binom{2n}{i-1}$.

We need the following theorem:

Theorem 2.4. [10] *For a graph* G *with even order* n *and no isolated vertices,* $\gamma(G) = \frac{n}{2}$ *if and only if the components of* G *are the cycles* C_4 *or the coronas* $H \circ K_1$ *for some connected graph* H*.*

Theorem 2.5. *If* G *is a graph of order* n*, then* $\gamma_a(G \circ K_1) = \gamma(G \circ K_1) + 1 = n + 1$*.*

Corollary 2.6. *Let* G *be a graph of order* n*.*

(i) For $n + 1 \leq m \leq 2n$*, we have* $d_a(G \circ K_1, m) = \binom{n}{m-n} 2^{2n-m}$*.*

(ii) $D_a(G \circ K_1, x) = x^n(x + 2)^n - 2x^n$*.*

Theorem 2.7. *(i) The number of accurate dominating sets of cycle* C_n *with cardinality* i *is*

$$d_a(C_n, i) = \left(\begin{array}{c} n \\ i \end{array} \right) - (n2^{n-i-3}).$$

(ii) The number of all accurate dominating sets of cycle C_n *is*

$$D_a(C_n, 1) = \sum_{i=\lfloor \frac{n}{3} \rfloor - \lceil \frac{3}{n} \rceil + 2} \left(\binom{n}{i} - n2^{n-i-3} \right).$$

Theorem 2.8. *The number of accurate dominating sets of path* P_n $(n \geq 7)$ *satisfies the following recurrence relation:*

$$d_a(P_n, i) = d_a(P_{n-1}, i - 1) + d_a(P_{n-2}, i - 1) + d_a(P_{n-3}, i - 1)$$

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On the Zagreb index in the fuzzy graph

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Abstract

In this paper, the Zagreb index in a fuzzy graph is investigated and formulas for the Zagreb indices of fuzzy graphs are also obtained. As a consequence of our results, some well-known assertions in the graph theory are obtained.

Keywords: fuzzy graph, fuzzy line graph, fuzzy common neighborhood graph, fuzzy Zagreb index.
Mathematics Subject Classification [2010]: Primary:05C72, 05C07; Secondary: 03E72.

1 Introduction

The fuzzy set theory proposed by Zadeh [10] is an extension of classical set theory. Zadeh's remarkable idea has found many applications in several fields, including the chemical industry, decision theory, networking, computer science, etc. Nowadays, many real-world problems cannot be properly modeled by a crisp graph theory, since the problems contain uncertain information. The fuzzy set theory, is used to handle the phenomena of uncertainty in real-life situation.

First we go through some basic definitions from [7, 8].

Definition 1.1. A fuzzy subset of a non-empty set S is a map $\sigma : S \to [0, 1]$ which assigns to each element x in S a degree of membership $\sigma(x)$ in $[0, 1]$ such that $0 \le \sigma(x) \le 1$. A fuzzy relation μ on S is a fuzzy subset of $S \times S$.

Let V be a nonempty set. Define the relation $\sim$ on $V \times V$ by for all $(x, y), (u, v) \in V \times V$, $(x, y) \sim (u, v)$ if and only if $x = u$ and $y = v$ or $x = v$ and $y = u$. Then it is easily shown that $\sim$ is an equivalence relation on $V \times V$. For all $x, y \in V$, let $[(x, y)]$ denote the equivalence class of (x, y) with respect to $\sim$. Then $[(x, y)] = \{(x, y), (y, x)\}$. Let $\mathcal{E}_V = \{[(x, y)] | x, y \in V, x \neq y\}$. For simplicity, we often write $\mathcal{E}$ for $\mathcal{E}_V$ when V is understood. Let $E \subseteq \mathcal{E}$. A graph is a pair (V, E). The elements of V are thought of as vertices of the graph and the elements of E as the edges. For $x, y \in V$, we let xy denote $[(x, y)]$. Then clearly $xy = yx$. We note that graph (V, E) has no loops or parallel edges.

Definition 1.2. A fuzzy graph $G = (V, \sigma, \mu)$ is a triple consisting of a nonempty set V together with a pair of functions $\sigma : V \to [0, 1]$ and $\mu : \mathcal{E} \to [0, 1]$ such that for all $x, y \in V$, $\mu(xy) \le \sigma(x) \wedge \sigma(y)$.

The fuzzy set σ is called the fuzzy vertex set of G and μ the fuzzy edge set of G.

Definition 1.3. Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The size of G is denoted by $S(G)$ and defined as $\sum\limits_{xy \in \mathcal{E}} \mu(xy)$. The degree $x \in V$ is denoted by $d_G(x)$ and defined as $d_G(x) = \sum\limits_{y \in V} \mu(xy)$.

[1] speaker

Let $G = (V, \sigma, \mu)$ be a fuzzy graph the neighbor of vertex v is denoted by $N_G(v)$ and is defined as follows:

$$N_G(v) = \{u \in V \mid \mu(uv) > 0\}.$$

Definition 1.4. The *fuzzy common neighborhood graph* or briefly *fuzzy congraph* of $G = (V, \sigma, \mu)$ is a fuzzy graph as $con(G) = (V, \omega, \lambda)$ such that $\omega(x) = \sigma(x)$ and

$$\lambda(uv) = min_{x \in H}\{\mu(ux).\mu(vx)\}$$

where $H = N_G(u) \cap N_G(v)$.

Definition 1.5. Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The *fuzzy line graph* of G is a fuzzy graph as $L(G) = (\mathcal{E}, \omega, \lambda)$ such that $\omega(e) = \mu(uv)$ for all $e = uv \in \mathcal{E}$ and $\lambda(e_1e_2) = \omega(e_1).\omega(e_2)$ for all $e_1 = uv_1$, $e_2 = uv_2$ in $\mathcal{E}$.

Two old and much studied degree–based graph invariants are the so-called *first and second Zagreb indices*, defined as

$$M_1(G) = \sum_{v \in V(G)} d^2(v) \qquad \text{and} \qquad M_2(G) = \sum_{uv \in E(G)} d(u)\, d(v)\ .$$

For details on their history, mathematical properties and chemical applications see [1, 3, 4] and the references cited therein. In the next definition, some new topological indices in a fuzzy graph are introduced which are natural generalizations of some well-known topological indices in the crisp graph, see [2, 5, 6, 9].

Definition 1.6. Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $Con(G) = (V, \omega, \lambda)$ its fuzzy congraph. We define *first and second fuzzy Zagreb indices*, *fuzzy forgotten topological index*, *first and second fuzzy common neighborhood indices* of G, respectively, as follows:

(1) $M_1^F(G) = \sum\limits_{v \in V} d_G^2(v);$

(2) $M_2^F(G) = \sum\limits_{uv \in \mathcal{E}(G)} \mu(uv) d_G(u) d_G(v);$

(3) $F^F(G) = \sum\limits_{v \in V} d_G^3(v);$

(4) $N_1^F(G) = \sum\limits_{v \in V} d_G(v) d_{con(G)}(v);$

(5) $N_2^F(G) = \sum\limits_{uv \in \mathcal{E}(Con(G))} \lambda(uv) d_G(u) d_G(v).$

2 Main Results

The next theorem characterize the degree of every vertex in the fuzzy line graph.

Theorem 2.1. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $L(G) = (\mathcal{E}, \omega, \lambda)$ its fuzzy line graph. Then*

$$d_{L(G)}(e) = \mu(v_iv_j)(d_G(v_i) + d_G(v_j) - 2\mu(v_iv_j)), \qquad e = v_iv_j \in \mathcal{E}(G).$$

From the above theorem, we can conclude the following result, which is trivial in the line graph.

Corollary 2.2. *Let $G = (V, E)$ be a graph and $L(G) = (E, W)$ the line graph of G. Then*

$$d_{L(G)}(e) = d_G(u) + d_G(v) - 2, \qquad e = uv \in E.$$

Lemma 2.3. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then, for every $v \in V$ the following holds:*

$$\sum_{v_iv_j \in \mathcal{E}} \mu(v_iv_j)(d_G^k(v_i) + d_G^k(v_j)) = \sum_{v_i \in V} d_G^{k+1}(v_i).$$

In particular,

$$\sum_{v_iv_j \in \mathcal{E}} \mu(v_iv_j)(d_G(v_i) + d_G(v_j)) = \sum_{v_i \in V} d_G^2(v_i) = M_1^F(G),$$

and

$$\sum_{v_iv_j \in \mathcal{E}} \mu(v_iv_j)(d_G^2(v_i) + d_G^2(v_j)) = \sum_{v_i \in V} d_G^3(v_i) = F^F(G).$$

Theorem 2.4. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $L(G) = (V, \omega, \lambda)$ its fuzzy line Graph. Suppose E is the fuzzy edge matrix of G, then*

$$2S(L(G)) = M_1^F(G) - 2tr(E \odot E).$$

Definition 2.5. Let $A = [a_{ij}]_{m \times n}$. Then, we define $S(A) = \sum_{1 \le i \le m,\ 1 \le j \le n} a_{ij}$.

Theorem 2.6. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $con(G) = (V, \omega, \lambda)$ its fuzzy congraph. Suppose that A_F, B_F, H_F and M_F are the fuzzy adjacency matrices of G and $con(G)$, the fuzzy vertices degree matrix and the fuzzy incidence matrix, respectively. Then*

1. $S(A_F) = 2S(G)$;
2. $S(A_F^2) = M_1^F(G)$;
3. $S(A_F H_F) = M_1^F(G)$;
4. $S(H_F A_F H_F) = 2M_2^F(G)$;
5. $S(A_F^3) = 2M_2^F(G)$;
6. $S(A_F B_F) = N_1^F(G)$;
7. $S(A_F B_F A_F) = 2N_2^F(G)$;
8. $S(H_F M_F) = M_1^F(G)$.

Theorem 2.7. *Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs such that $n_1 = |V_1|$, $n_2 = |V_2|$ and $k = \min\{\sigma_1(u), \sigma_2(v)\}$ for all $u \in V_1$ and $v \in V_2$. Then*

1. $S(G_1 \vee G_2) = S(G_1) + S(G_2) + n_1n_2k$;
2. $M_1^F(G_1 \vee G_2) = M_1^F(G_1) + M_1^F(G_2) + n_1n_2k^2(n_1 + n_2) + 4k(n_2S(G_1) + n_1S(G_2))$;
3. $M_2^F(G_1 \vee G_2) = M_2^F(G_1) + M_2^F(G_2) + k^2(n_2^2S(G_1) + n_1^2S(G_2)) + k(n_2M_1^F(G_1) + n_1M_1^F(G_2)) + 2n_1n_2k^2(S(G_1) + S(G_2)) + 4kS(G_1)S(G_2) + n_1^2n_2^2k^2$.

Theorem 2.8. *Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs such that $n_1 = |V_1|$, $n_2 = |V_2|$. Then*

1. $S(G_1 \times G_2) = n_2S(G_1) + n_1S(G_2)$;
2. $M_1^F(G_1 \times G_2) = n_2M_1^F(G_1) + n_1M_1^F(G_2) + 8S(G_1)S(G_2)$;
3. $M_2^F(G_1 \times G_2) = 3S(G_2)M_1^F(G_1) + 3S(G_1)M_1^F(G_2) + n_1M_2^F(G_2) + n_2M_2^F(G_1)$.

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A neighborhood condition and characteristic polynomials of some graphs

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Abstract

A simple graph is called N-bounded if for every two nonadjacent vertices x, y, there exists a vertex z such that $N(x) \cup N(y) \subseteq N(z) \cup \{z\}$. It is shown that any regular graph is complete r-partite with parts of the same size, where r is the number of the connected components of $\overline{G}$ and then its characteristic polynomial is computed. Also, the structure and the adjacency spectrum of N-bounded bipartite graphs are studied. Finally the number of nonzero eigenvalues of an N-bounded bipartite graph is determined.

Keywords: Characteristic polynomial, eigenvalue, N-bounded graph, Neighborhood.

Mathematics Subject Classification [2010]: Primary: 05C31, 05C50; Secondary: 05C75

1 Introduction

Throughout this paper, all graphs are simple graphs (i.e. undirected graphs without loops and multiple edges). For any graph G, the vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. Here, $\overline{G}$ denotes the *complement* of G. By K_n and C_n, we mean a *complete graph* with n vertices and a *cycle* with n vertices, respectively. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If for any $x, y \in V(H)$, $xy \in E(G)$ implies $xy \in E(H)$, then H is called an *induced subgraph* of G; moreover, if $V(H) = X$, then the induced subgraph H is denoted by $G[X]$. The *neighborhood* of a vertex x is $N_G(x) = \{y \in V(G)|\ xy \in E(G)\}$, the size of $N_G(x)$, denoted by $deg_G(x)$, is the *degree* of x. The set $N_G[x] = N_G(x) \cup \{x\}$ is called the *closed neighborhood* of x. When there is no ambiguity, subscripts can be omitted. A vertex with degree 0 (1) is called an *isolated* (*pendant*) vertex. For any graph G, all end vertices which are adjacent to a same vertex of G together with the edges is called a *horn*. If every vertex of G has degree k, then G is said to be *k-regular*. An *r-partite* graph is a graph whose vertex set can be partitioned into r subsets so that no edge has both ends in the same subset. A *complete r-partite* graph is an r-partite graph in which each vertex is adjacent to every vertex that is not in the same part. In particular, a (complete) 2-partite graph is called (complete) *bipartite*. A complete bipartite graph one of whose parts has size one is called a *star* graph. Also, by a *refinement of a star graph*, we mean a graph which contains a vertex that is adjacent to every other vertices. By eigenvalues of the graph G, we mean the eigenvalues of its adjacency matrix $A = A(G)$. Also, in this paper, the characteristic polynomial of a graph G is defined to be the characteristic polynomial of its adjacency matrix which is denoted by $\chi(G, \lambda)$ For more details and the undefined concepts, see [1, 2]. A graph G is called *N-bounded* if G contains no isolated vertices and for each pair x, y of nonadjacent vertices of G, there is a vertex z with $N(x) \cup N(y) \subseteq N[z]$. In this paper, the adjacency spectrum of two classes of N-bounded graphs (regular graphs and bipartite graphs) is studied.

[1]speaker

2 Main Results

We start with the following theorem which determines when the join of a family of graphs is N-bounded?

Theorem 2.1. *Let* $\{G_\alpha\}_{\alpha\in\Lambda}$ *be a family of graphs. Then* $G = \bigvee_{\alpha\in\Lambda} G_\alpha$ *is an N-bounded graph if and only one of the following conditions holds:*

(1) G_α *is the refinement of a star graph, for some* $\alpha \in \Lambda$.

(2) For every $\alpha \in \Lambda$, *there exists a non-negative integer* n_α *and an N-bounded graph* H_α *such that* $G_\alpha = H_\alpha \vee K_{n_\alpha}$.

Note that in the previous, maybe either $n_\alpha = 0$ or $H_\alpha = \varnothing$ which implies that G_α is either an N-bounded graph or a null graph. Also, by using Theorem 2.1, one can prove the following results.

Proposition 2.2. *If* G *is an N-bounded regular graph, then* $\overline{G}$ *is a disconnected graph.*

The previous proposition, in fact, says that the complement of any regular N-bounded graph has at least two connected components. This fact can be used to study the structure of regular N-bounded graphs.

Theorem 2.3. *Let* G *be an* r*-regular N-bounded graph of order* n. *Then* G *is a complete* $c(\overline{G})$*-partite graph whose parts have the same cardinality.*

Theorem 2.4. (See [2]) *Let* $G = K_{n_1,n_2,\ldots,n_r}$ *be a complete* r*-partite graph. The the characteristic polynomial of* G *is*

$$\chi(G,\lambda) = \lambda^{n-r}\left(1 - \sum_{i=1}^{r}\frac{n_i}{\lambda + n_i}\right)\prod_{j=1}^{r}(\lambda + n_j).$$

From Theorems 2.3 and 2.4, we have the following immediate corollary which determines the adjacency spectrum of regular N-bounded graphs.

Corollary 2.5. *If* G *is a regular N-bounded graph whose complement has* r *connected components, then*

$$\chi(G,\lambda) = \lambda^{n-r}\left(1 - \frac{rn}{r\lambda + n}\right)\left(\lambda + \frac{n}{r}\right)^r.$$

In the continuing, the adjacency spectrum of triangle-free N-bounded graphs triangle-free N-bounded graphs are studied. In [3], it has been proved that any vertex N-bounded graph with degree at least two lies on an either triangle or square.

Theorem 2.6. *Any non-pendant vertex of an N-bounded graph belongs to a cycle of length at most four.*

Proof. This follows from [3, Theorem 14]. □

To compute the characteristic polynomial of triangle-free N-bounded graphs, recalling the following result from "Algebraic Graph Theory" is needed.

Theorem 2.7. (See [2, Page 78]) *Let* G *be a simple graph and for every vertex* v *of* G, $\mathcal{C}(v)$ *denote the set of all cycles in* G, *containing* v. *Then*

$$\chi(G,\lambda) = \lambda\chi G - v,\lambda) - \sum_{uv\in E(G)}\chi(G - v - u,\lambda) - 2\sum_{C\in\mathcal{C}(v)}\chi(G - V(C),\lambda).$$

Recall that a *two-star graph* is a graph G consisting of two star graphs with a bridge connecting the two sub-centers x and y, which can be drawn as $U — x — y — V$, where $|U| \geq 1$, $|V| \geq 1$. If $|U| = t$ and $|V| = k$, then the two-star graph is denoted by $S_{t,k}$.

Theorem 2.8. *The characteristic polynomial of the two-star graph* $S_{t,k}$ *equals:*

$$\chi(S_{t,k},\lambda) = \lambda^{t+k+2} - k\lambda^{t+k} - t\lambda^{t+k} + tk\lambda^{t+k-2} - \lambda^{t+k}.$$

The following result shows that any triangle-free N-bounded graph is bipartite.

Proposition 2.9. (See [3, Proposition 5]) *An N-bounded graph is bipartite if and only if it is triangle-free.*

Notation. By $K_{p,q}^{m,n}$, we denote a graph which is obtained from the complete bipartite graph $K_{p,q}$ after attaching a horn with m vertices on one of the parts and a horn with n vertices on the other part (see Figure 1).

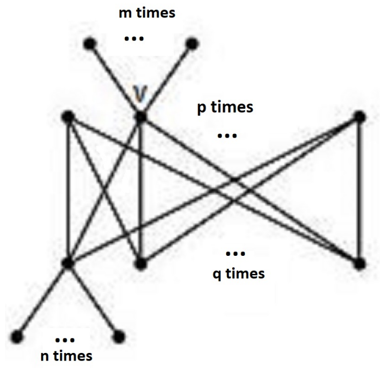

Figure 1: The graph $K_{p,q}^{m,n}$, $m \geq 0, n \geq 0$

Theorem 2.10. *A bipartite graph G is N-bounded if and only if $G \cong K_{p,q}^{m,n}$, for some non-negative integers m, n and positive integers p, q.*

Proof. This follows from [3, Theorem 11]. □

Remark 2.11. Using counting principles, one can easily check that the complete bipartite graph $K_{p,q}$ contains $\binom{p-1}{k-1}\binom{q}{k}\frac{k}{2}\left[(k-1)!\right]^2$ distinct cycles of length $2k$.

Lemma 2.12. *The characteristic polynomial of the bipartite graph $G = K_{p,q}^{m,0}$ is*

$$\begin{aligned}\chi(G,\lambda) &= \lambda^{m+p+q-2}(\lambda^2 - q(p-1)) - q\lambda^{m+p+q-4}(\lambda^2 - (p-1)(q-1)) - m\lambda^{m+p+q-4}(\lambda^2 - q(p-1)) \\ &- 2\sum_{i=1}^{p-1}\binom{p-1}{i}\binom{q}{i+1}\frac{i+1}{2}(i!)^2\lambda^{m+p+q-2i-4}(\lambda^2 - (p-i-1)(q-i-1)).\end{aligned}$$

From Lemma 2.12, we have the following immediate corollary.

Corollary 2.13. *In the adjacency spectrum of the bipartite graph $G = K_{p,q}^{m,0}$, there are exactly 4 nonzero eigenvalues.*

Finally, by using Lemma 2.12 and a simple computation, the following result is proved.

Theorem 2.14. *Any N-bounded bipartite graph with $n \geq 6$ vertices, has 6 nonzero eigenvalues in its adjacency spectrum.*

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l-metric generator in graphs and its application

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Abstract

In this paper, by applying the notion of metric dimension of a graph, we introduce a unique code for each vertex of a given graph such that the length of codes be as small as possible and each vertex can be identified by its code. Indeed, for an ordered subset $S = \{v_1, \ldots, v_k\}$ of vertices in a connected graph G and a vertex v of G, the *l-metric S-representation* of a vertex $v \in V(G)$ is the vector $r_G^l(v|S) = (a_G(v, v_1), \ldots, a_G(v, v_k))$, where $a_G(v, v_i) = \min\{d_G(v, v_i), l\}$, $i \in \{1, \ldots, k\}$. $S \subseteq V(G)$ is a *l-metric generator* for G if the vertices of G have pairwise different l-metric S-representations. A l-metric generator of smallest order is a *l-metric basis* for G, its order being the *l-metric dimension* $\dim_l(G)$ of G. We prove that the length of codes obtained from metric dimensions for some graphs is larger than the length of codes obtained from our new definition of metric dimensions and the difference can be large unbounded.

Keywords: Graph, metric dimension; l-metric dimension.

AMS Mathematical Subject Classification [2010]: Primary: 05C12, 05C85; Secondary: 90B80

1 Introduction

All graphs considered in this paper are connected and simple. If $G = (V(G), E(G))$ is a graph, then its order and its size are denoted with $n(G)$ and $m(G)$, respectively. If $u, v \in V(G)$, then $d_G(u, v)$ denotes the standard shortest-path distance between u and v in G, that is, the number of edges on a shortest u, v-path. If $S = \{v_1, \ldots, v_k\} \subseteq V(G)$, then the *metric S-representation* of a vertex $v \in V(G)$ is the vector

$$r_G(v|S) = (d_G(v, v_1), \ldots, d_G(v, v_k)) .$$

$S \subseteq V(G)$ is a *metric generator* for G if the vertices of G have pairwise different metric S-representations. A metric generator of smallest order is a *metric basis* for G, its order being the *metric dimension* $\dim(G)$ of G.

The metric dimension was introduced in [4, 8] and is used to model many real world problems such as navigation of robots in networks [6] and chemical problems [5]. However, quite often we need to distinguish all pairs of vertices by sequences not by vectors. Note that, the length of a vector is the number of its arrays while the length of a sequence is the number of its figures. For more explanation, assume that a bank decides to design codes which show both the customers' account number and address. In this case, when

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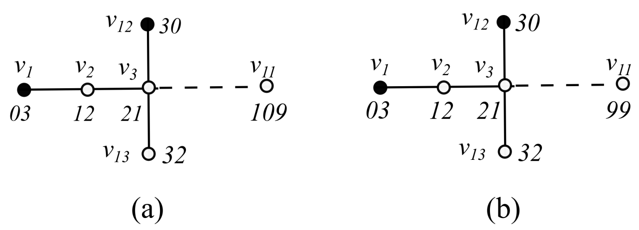

Figure 1: Earlier codes.b) Present codes.

the customers enter their account details in a shopping cart's homepage, not only they will pay for their online shopping but also give their delivery address simultaneously. For more illustration, let Figure 2 show the customer position map of a bank. To design a graph theory model for this problem, consider customers as the vertices of a graph Γ, and connect two vertices c_i and c_j in Γ whenever there is no other customer on the c_i, c_j-geodesics. Let S be a local metric basis for Γ. In this case, if we consider codes obtained by the classic form of mertric generator, costumers have account details with different length. Indeed, as it is seen in Table 1, the sequences produced by the classic method do not necessarily have the same length. Then, the question arises whether a bank can obtain the codes with the same length, because customers' account details should be constructed by the same number of digits. The ansewer to this question channel us towards introducing a new concept which is called the l-metric generator.

If $S = \{v_1, \ldots, v_k\} \subseteq V(G)$, then the *l-metric S-representation* of a vertex $v \in V(G)$ is the vector

$$r_G^l(v|S) = (a_G(v, v_1), \ldots, a_G(v, v_k)),$$

where $a_G(v, v_i) = \min\{d_G(v, v_i), l\}$, $i \in \{1, \ldots, k\}$. $S \subseteq V(G)$ is a *l-metric generator* for G if the vertices of G have pairwise different l-metric S-representations. A l-metric generator of smallest order is a *l-metric basis* for G, its order being the *l-metric dimension* $\dim_l(G)$ of G. Note that our definition of l-metric dimension is different from what was studied in [2, 3, 7]. In this paper, we focus on 9-metric generators. As an important result of this concept, we can point out to producing codes with the same length by 9-metric generator. For example see Figure 1. In part (a) of this figure, length of v_{11}'s code obtained by metric $\{v_1, v_{12}\}$-representation is three, while length of other vertices; codes is equal to 2. But in part (b), length of each vertex's code obtained by 9-metric $\{v_1, v_{12}\}$-representation is equal to 2. Another interesting point in Figure 1 is that the length of v_{11}'s code decreases by 1 to 2 in part (b).

In the next section we give a new model for assigning codes to bank's customers. The model uses l-metric bases and is in many cases significantly more efficient than the classical model form [6].

2 Main Results

In this section, we compare the efficiency of the new version of metric generator set which is introduced in this paper compared to the previous version. To do this, remember the example related to producing customers account number which is presented in the previous section.

Table 1 provides some information on codes of customers c_8, c_{10}, c_{55} which are obtained by earlier metric representation and l-metric representation. As the table shows $dim(\Gamma) = dim_9(\Gamma)$, while the length of codes obtained by 9-metric representation are shorter than ones obtained by earlier metric representation. Also, unlike the codes presented by earlier metric representation, 9-metric representation' codes have the same length.

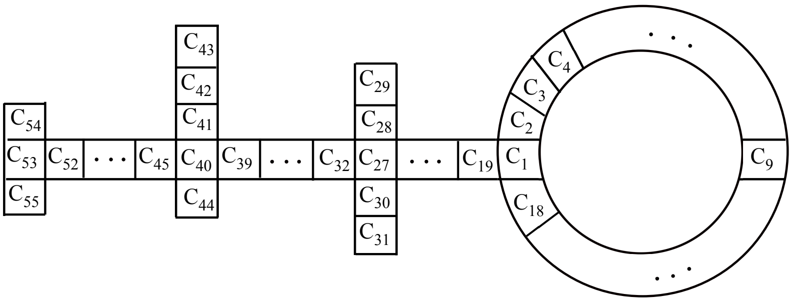

Figure 2: The leyout of customers' position.

Figure 3: The graph Γ corresponding to the plan of Figure 1 (the black vertices form a metric basis and a 9-metric basis for Γ).

Table 1: Earlier and present codes of some vertices of G.

Vertices	c_8	c_{10}	c_{55}
Earlier metric representation	$(7, 1, 18, 26, 35)$	$(9, 1, 20, 28, 37)$	$(36, 28, 21, 11, 2)$
Earlier codes (customer's account details)	71182635	91202837	362821112
9-Metric representation	$(7, 1, 9, 9, 9)$	$(9, 1, 9, 9, 9)$	$(9, 9, 9, 9, 2)$
Present codes (customer's account details)	71999	91999	99992

By the below proposition, we show that, in plenty of cases, the codes formed by 9-metric representation is significantly shorter than the earlier codes.

Theorem 2.1. *Let T_n be a tree with the central path $v_1v_2\ldots,v_n$ such that $T_n - v_i$, $i \in \{1, 2, \ldots, n\}$, is a forest with more that three components. Then*

$$k - t \geq n^2 - 17n + 36,$$

where k and t are the length of codes obtained by metric representation and 9-metric representation, respectively.

Acknowledgment

The acknowledgements should be in a separate section at the end of the article before the references.

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On the Generalized Regular Covering of Graphs with respect to a Subgroup

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Abstract

In this paper, by reviewing the concept of regular and $H-$regular covering of graphs, we extend properties of regular covering of graph to properties of $H-$regular covering of graph. For example, by definition of $Cov_{H,\widetilde{g_0}}(\widetilde{G}/G)$, we obtain $Cov_{H,\widetilde{g_0}}(\widetilde{G}/G) \cong H/p_*\pi_1(\widetilde{G},\widetilde{g})$, where $p : \widetilde{G} \to G$ is an $H-$regular covering of graph and $H \leq \pi_1(G, g_0)$.

Keywords: graph, fundamental group, covering of graph.
Mathematics Subject Classification [2010]: Primary: 57M10, 57M12; Secondary: 57M05

1 Introduction

Let $\tilde{G} = (V_1, E_1)$ and $G = (V_2, E_2)$ be two graphs, and let $p : V_1 \to V_2$ is a surjection. Recall that p is a covering map from $\tilde{G}$ to G if for each $v \in V_1$, the restriction of p to the neighborhood of v is a bijection onto the neighborhood of $p(v)$ in G. Put otherwise, p maps edges incident to v one-to-one onto edges incident to $p(v)$.

Since π_1 is a (covariant) functor, $\pi_1(p) = p_* : \pi_1(\tilde{G}, \tilde{g}) \to \pi_1(G, g)$ is a homomorphism, where $\tilde{g} \in \tilde{G}$ and $g \in G$. If we study p_*, then we can obtain more information about p (see [1, 2]). We recall that $p : \widetilde{G} \to G$ is a regular covering of graph, if $p_*\pi_1(\widetilde{G})$ is a normal subgroup of $\pi_1(G)$ and let $H \leq \pi_1(G, g_0)$, a covering map $p : \widetilde{G} \to G$ is an $H-$regular if $p_*\pi_1(\widetilde{G}, \widetilde{g})$ is a normal subgroup of H, for all $\widetilde{g} \in p^{-1}(g_0)$. For example, every covering map $p : \widetilde{G} \to G$ is an $\{e\}-$regular. Note that $\pi_1(\tilde{G}, \tilde{g_0})$ acts on $p^{-1}(g_0)$ and this property has interesting results (see [3, 4, 5]). We recall that $\widetilde{g} \sim_H \widetilde{y}$ if and only if there is an $h \in H$ such that $h\widetilde{g} = \widetilde{y}$ i.e. $\tilde{h}(1) = \widetilde{y}$ where $\tilde{h}$ is the liffting of h at $\widetilde{g}$. Note that $\sim_H$ is an equivalence relation on $p^{-1}(g_0)$. For every $\widetilde{g_0} \in p^{-1}(g_0)$ the equivalence class $[\widetilde{g_0}]$ under relation $\sim_H$ denoted by $p^{-1}_{H,\widetilde{g_0}}(g_0)$. Infact $p^{-1}_{H,\widetilde{g_0}}(g_0) = \{\tilde{g} \in p^{-1}(g_0) | \tilde{g} \sim_H \widetilde{g_0}\}$ and $Cov_{H,\widetilde{g_0}}(\widetilde{G}/G) = \{f : \widetilde{G} \to \widetilde{G}$ *is an isomorphism of graph such that* $p \circ f = p$ *and* $f(p^{-1}_{H,\widetilde{g_0}}(g_0)) = p^{-1}_{H,\widetilde{g_0}}(g_0)\}$.

Remark 1.1. If $H = \pi_1(G, g_0)$, then $p^{-1}_{H,\widetilde{g_0}}(g_0) = p^{-1}(g_0)$ and $Cov_{H,\widetilde{g_0}}(\widetilde{G}/G) = Cov(\widetilde{G}/G)$.

By Remark 1.1, The following lemma is an extension of [3, Theorem 10.9].

Lemma 1.2. *Let* $p : \widetilde{G} \to G$ *be a covering of graph, let* $g_0 \in G$*, and let* $H \leq \pi_1(G, g_0)$*.*

1. *H acts transitively on* $p^{-1}_{H,\widetilde{g_0}}(g_0)$*;*
2. *If* $\widetilde{G} \in p^{-1}_{H,\widetilde{g_0}}(g_0)$*, then the stabilizer of* $\widetilde{G}$ *is* $p_*\pi_1(\widetilde{G}, \widetilde{g})$*;*
3. $|p^{-1}_{H,\widetilde{g_0}}(g_0)| = [\pi_1(G, g_0) : p_*\pi_1(\widetilde{G}, \widetilde{g_0})]$.

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2 $H-$regular covering of graph

The following lemma is a generalization of [3, Lemma 10.24] by Remark 1.1 (for graphs).

Lemma 2.1. *Let $p : \widetilde{G} \to G$ be a covering of graph, let $g_0 \in G$ and recall that $p^{-1}_{H,\widetilde{g_0}}(g_0)$ is a transitive H-set, where $H \leq \pi_1(G, g_0)$. Given $\widetilde{g}, \widetilde{y} \in p^{-1}_{H,\widetilde{g_0}}(g_0)$, there exists $f \in Cov_{H,\widetilde{g_0}}(\widetilde{G}/G)$ with $f(\widetilde{g}) = \widetilde{y}$ if and only if there exists $\phi \in Aut(p^{-1}_{H,\widetilde{g_0}}(g_0))$ with $\phi(\widetilde{g}) = \widetilde{y}$.*

Since $Cov_{H,\widetilde{g_0}}(\widetilde{G}/G) \subseteq Cov(\widetilde{G}/G)$ and By [3, Theorem 10.19], we can conclude following lemma.

Lemma 2.2. *Let $p : \widetilde{G} \to G$ be a covering of graph.*

1. *If $f \in Cov_{H,\widetilde{g_0}}(\widetilde{G}/G)$ and $f \neq 1_{\widetilde{g}}$, then f has no fixed points.*
2. *If $f_1, f_2 \in Cov_{H,\widetilde{g_0}}(\widetilde{G}/G)$ and there exists $\widetilde{g} \in \widetilde{G}$ with $f_1(\widetilde{g}) = f_2(\widetilde{g})$, then $f_1 = f_2$.*

By Remark 1.1, the following theorem is an extension of [3, Lemma 10.25] (for graphs).

Theorem 2.3. *Let $p : \widetilde{G} \to G$ be a covering of graph, let $g_0 \in G$ and let the set $p^{-1}_{H,\widetilde{g_0}}(g_0)$ be viewed as H-set, where $H \leq \pi_1(G, g_0)$. Then $f \longmapsto f|_{p^{-1}_{H,\widetilde{g_0}}(g_0)}$ is an isomorphism*

$$Cov_{H,\widetilde{g_0}}(\widetilde{G}/G) \cong Aut(p^{-1}_{H,\widetilde{g_0}}(g_0))$$

The following theorem is a generalization of [3, Theorem 10.27] (see Remark 1.1)(for graphs).

Theorem 2.4. *Let $p : \widetilde{G} \to G$ be a covering of graph. If $g_0 \in G$, $H \leq \pi_1(G, g_0)$ and $\widetilde{g} \in p^{-1}_{H,\widetilde{g_0}}(g_0)$, then*

$$Cov_{H,\widetilde{g_0}}(\widetilde{G}/G) \cong N_H(p_*\pi_1(\widetilde{G}, \widetilde{g}))/p_*\pi_1(\widetilde{G}, \widetilde{g}).$$

The following corollary is a consequence of above theorem.

Corollary 2.5. *Let $p : \widetilde{G} \to G$ be an H-regular covering of graph and $H \leq \pi_1(G, g_0)$. Then, for $g_0 \in G$ and $\widetilde{g} \in p^{-1}_{H,\widetilde{g_0}}(g_0)$,*

$$Cov_{H,\widetilde{g_0}}(\widetilde{G}/G) \cong H/p_*\pi_1(\widetilde{G}, \widetilde{g}).$$

The following corollary is a consequence of above theorem.

Corollary 2.6. *Let $p : \widetilde{G} \to G$ be an universal covering of graph and $H \leq \pi_1(G, g_0)$. Then, for $g_0 \in G$,*

$$Cov_{H,\widetilde{g_0}}(\widetilde{G}/G) \cong H.$$

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Extra connectivity of lexicographic product graphs

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Abstract

Lexicographic product of two graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, denoted by $G_1 \times G_2$, has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \times G_2) = \{(a, b)(x, y) \mid ax \in E(G_1) \ \ or \ \ a = x, by \in E(G_2)\}$. A vertex-cut S of a graph G is called an e-extra vertex-cut if $G - S$ is disconnected and every componrnt of $G - S$ has more than e vertices. The e-extra connectivity of a graph G, $\kappa_e(G)$, is defined as the minimum cardinality over all e-extra vertex-cuts of G. In this paper we investigate e-extra connectivity of $G_1 \times G_2$ and give some results.

Keywords: connectivity, extra connectivity, lexicographic product

Mathematics Subject Classification [2010]: 05C40, 05C82, 05C25

1 Introduction

We follow [2] for graph theoretic terminologies and notations not defined here. In this paper, a graph $G = (V(G), E(G))$ means an undirected simple graph. For $x \in V(G)$, $N_G(x)$ denotes the set of neighbors of x in G. Also, $deg_G(x) = |N_G(x)|$. Let K_n and C_n denote the complete graph and the cycle graph with n vertices, respectiviely.

The lexicographic product has an important role in the analysis of network, see [10] and [1]. Lexicographic product of two graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, denoted by $G_1 \times G_2$, has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \times G_2) = \{(a, b)(x, y) \mid ax \in E(G_1) \ \ or \ \ a = x, by \in E(G_2)\}$. It is easy to check that $G_1 \times G_2$ is connected if and only if G_1 is connected. For more results of the connectivity of $G_1 \times G_2$, see [1], [3]– [4], [7] and [12].
A set $S \subset V(G)$ is called a vertex-cut of G if either $G - S$ is disconnected or reduces to the trivial graph K_1. For a vertex-cut S of a graph G, S_0 is the set of all $u \in V(G)$ such that $N_G(u) \subseteq S$. A vertex-cut S is called super vertex cut if $G - S$ contains no isolated vertices. Also, a vertex-cut S is an e-extra vertex-cut if $G - S$ is disconnected and every componrnt of $G - S$ has more than e vertices. The connectivity of a graph G, $\kappa(G)$, is defined as the minimum cardinality over all vertex-cuts of G and is an important measure of network reliability. The super connectivity of a graph G, $\acute{\kappa}(G)$, is defined as the minimum cardinality over all super vertex-cuts of G. If $\acute{\kappa}(G)$ is not exist then we write $\acute{\kappa}(G) = \infty$. It is easy to check that $\acute{\kappa}(K_{1,n}) = \infty$. Furthermore, the e-extra connectivity of a graph G, $\kappa_e(G)$, is defined as the minimum cardinality over all e-extra vertex-cuts of G.

The connectivity and super connectivity of lexicographic product graphs has been studied in [7] and [12]. For more results of super connectivity of product graphs , see [5]– [9] and [11]. In this paper we investigate the e-extra connectivity of lexicographic product graphs.

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2 extra connectivity

It was shown that if G is a connected graph then $G \times H$ is also a connected graph. Also, $\kappa(K_n) = n - 1$. So, in all results we consider G as a connected non complete graph.

Theorem 2.1. *Let G be a connected non complete graph with $\kappa(G) < \acute{\kappa}(G) = \infty$. If H is a totally disconnected graph, then $\kappa_e(G \times H) = \infty$ for $e \geq 1$.*

Theorem 2.2. *Let G be a connected non complete graph with $\kappa(G) < \acute{\kappa}(G) = \infty$. If H is a disconnected graph with no isolated vertices, then $\kappa_e(G \times H) = \kappa(G)|H|$ for $e < |\acute{H}|$ where $\acute{H}$ is the smallest component of H.*

Theorem 2.3. *Let G be a connected non complete graph with $\kappa(G) < \acute{\kappa}(G) = \infty$. If H is a disconnected graph with $V_0(H) \neq \emptyset$, then $\kappa_e(G \times H) = \kappa(G)|H| + t$ for $e < |\acute{H}|$ where $\acute{H}$ is the smallest component of H and $t = |V_0(H)||S_0|$.*

Theorem 2.4. *Let G be a connected non complete graph with $\kappa(G) = \acute{\kappa}(G)$ and S be a super vertex-cut of G with minimum size . If H is a graph, then $\kappa_e(G \times H) = \kappa(G)|H|$ for $e < |H|t$ where t is the size of smalles component of $G - S$.*

Corollary 2.5. *Let G be a connected non complete graph with $\kappa(G) < \acute{\kappa}(G) < \infty$ and S be a super vertex-cut of G with minimum size . If H is a graph, then $\kappa_e(G \times H) = \acute{\kappa}(G)|H|$ for $e < |H|t$ where t is the size of smalles component of $G - S$.*

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Sequence pairs related to produced graphs by a method for dividing a natural number by two

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Abstract

This paper is about producing of a new kind of the pairs which we call it MS-pairs. To produce these pairs, we use an algorithm for dividing a natural number x by two for two arbitrary numbers and consider their related graphs. Because of the interesting properties of the MS-pairs, such as unpredictability, irreversible, aperiodicity and chaotic behavior, we think they are useful and applicable in cryptography, chaos theory, random number generation, stegnaography, password hashing, and unique identifier generators. All of the concepts, implemented with python 3.7.

Keywords: Diamond, DGBT, MZ-Algorithm
Mathematics Subject Classification [2010]: Primary: 22D15;; Secondary: 43A10, 43A20

1 Introduction

In [1] a new method (which is MZ-algorithm), has presented for dividing a natural number x by two and used graphs as models to show MZ-algorithm. Applying k-times of the MZ-method for the number x, creates a graph with unique structure that is denoted by $G_k(x)$ and is called DGBT. It is easy to see that $G_k(n)$ is not tree for $k > 1$, since the graph has cycle. See the graph $G_2(375)$ in Figure 1 (see [1]). It is easy to see that the number of cycles C_8

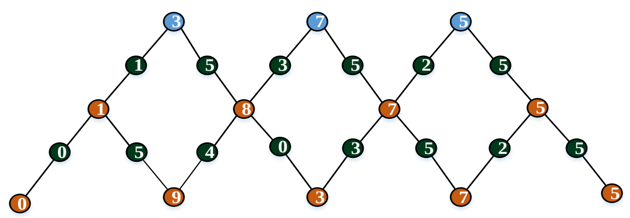

Figure 1: The graph $G_2(375)$.

in the graph $G_k(x)$ is $\frac{k-1}{2}(2d + k - 2)$. We show the figures of these cycles C_8 in $G_k(x)$ similar to diamond. After finding all diamonds in the graph $G_k(x)$, we label them by number $0, 1, 2, \ldots$ and write these label inside diamond (Figure 2).

We consider these diamond structures and use it to produce a sequence of pairs, that we call these pairs MS-pairs.

[1]speaker

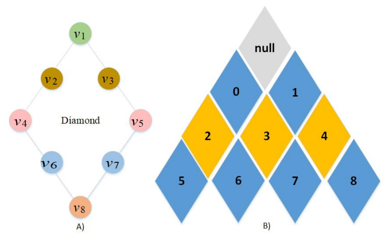

Figure 2: A) diamond structure. B) This picture shows nine diamonds related to level 1 to 11 in DGBT.

2 MS-pairs Generation

Definition 2.1. Consider the diamond d_i from the graph $G_k(x_1)$, and the diamond d_j from the graph $G_k(x_2)$. We say that these two diamonds are equal, if all values of vertices in the diamond d_i are equal to all values of vertices in the diamond d_j. Note that we compare the value of a vertex in d_i with the value of the same vertex in diamond d_2.

Definition 2.2. If the diamond d_i from graph $G_k(x_1)$ is equal to diamond d_j from graph $G_k(x_2)$, then we consider the label of these two diamonds as the first and the second component of a pair, i.e., (i, j). We call this pair, a MS-pair.

Observed that by considering the graphs $G_k(x_1)$ and $G_k(x_2)$ and applying MZ-algorithm k-times (for large enough number k) we can produces sequences of MS-pairs that because of the interesting properties of these pairs (such as unpredictability, irreversible, aperiodicity and chaotic behavior) we think they are useful and applicable in cryptography, chaos theory, random number generation, stegnaography, password hasshing, and unique identifier generators. Note that if we have all of the MS-pairs that generated from $G_k(x), G_k(y)$, we cannot predict or determine the root numbers of graphs, i.e. x and y.

The following algorithm, describe the MS-pairs generation steps in detail.

Algorithm 1: Pairs Generation Algorithm

```
input  : G_k(x_1), G_k(x_2) two different DGBT
output: Sequence of pairs(MS − pairs)
while Dimond in G_k(x_1) do
    d_i ← (Select a non − zero diamond from G_k(x_1));
    while Dimond in G_k(x_2) do
        d_j ← (Select a non − zero diamond from G_k(x_2);
        if(d_i == d_j)
        add to sequence of pairs(i, j);
Output ← Sequence of pairs ;
```

The following are some properties of MS-pairs:

Theorem 2.3. *(i) If d is the number of diamonds in the DGBT graph $G_l(n)$, then*

$$d = \binom{l+1}{2} - \binom{n}{2}.$$

(ii) If $G_k(x)$ is an infinite DGBT, then the number of its diamonds is infinite.

(ii) MS-pairs are unique. In other words, there is no two numbers x and y such that MS-pairs produced by $G_k(x)$ and $G_k(y)$ are exactly the same.

3 Brief review on applications of the MS-pairs

As the first example for the application of MS-pairs, consider two graphs $G_k(x)$ and $G_k(y)$, where x is a number that we can consider it as a time (which is a number with at most digit, for example the time 7:19:27 consider as number 71927) and y is an arbitrary constant number. We have shown the DGBT for the time 7:19:27 in Figure 3. It is interesting that if the number related to time changed, and the number y does not change, then the MS-pairs will be changed.

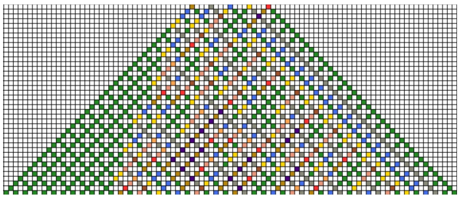

Figure 3: Time based DGBT of time 7:19:27.

Example 3.1. Suppose that $x = 7 : 19 : 27$ and $y = 45218$. We have 11438 pairs until level 100, i.e., in $G_{100}(x)$ and $G_{100}(y)$. We bring up some of these MS-pairs in the following:

$\{(46, 452), (58, 274), (58, 484), (71, 287), (71, 365), (71, 483), (84, 312), (85, 516), (95, 2139), ...\}$.

Now if we change time x to $x_1 = 71928$ we have 11136 pairs in $G_{100}(x_1)$ and $G_{100}(y)$. The following are some of these MS-pairs:

$\{(46, 452), (58, 484), (71, 287), (71, 365), (71, 483), (80, 1765), (84, 312), (85, 516), (90, 136), ...\}$.

We compare these two sequences of pairs. Only the pairs $(46, 452)$, $(84, 312)$, $(85, 516)$ are equal from these two sequence of MS-pairs. As we see, a small change in the root number, causes very large changes in the correspond MS-pairs. This phenomenon is referred to butterfly effect in chaos theory [2]. Also, we can use time based DGBT in Non-repudiation applications. As an example, assume that two persons A, B claim that in the time t, signed the contract d. To prove their claims, we can produce MS-pairs for A, B (for example with their ID-numbers) with t. If the produced MS-pairs are equal, then the claim of that person is true.

3.1 Applications to Steganography and Cryptography

Steganography is the practice of concealing a message within another message or a physical object [3]. For concealing the message M on the file X, we can use MS-pairs.

Stream cipher is an important branch of symmetric cryptosystems, which takes obvious advantages in speed and scale of hardware implementation. It is suitable for using in the cases of massive data transfer or resource constraints, and has always been a hot and central research topic in cryptography. A word-oriented stream cipher usually works on and outputs words of certain size, like 32, 16, 8 bits [4]. We will use MS-pairs for word-stream-cipher cryptographic application, word size in this algorithm is 8 bits. We present algorithms to show that how MS-pairs could to encrypt and decrypt data or message.

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On a Class of Symplectic Graphs and Their Automorphisms

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Abstract

It easy to see that each graph is a modification of a reduced graph Γ of the same rank. It is proved that for every reduced graph with binary rank $2r$, there is a unique maximal graph with binary rank $2r$ which conatins Γ as an induced subgraph. These maximal graphs are called symplectic graphs. In this paper, we study the symplectic graphs which are defined over a ring. We also find the automorphism group of symplectic graphs which are defined over $\mathbb{Z}_{p^n}$, where p is a prime number and n is positive integer.

Keywords: Automorphism, Symplectic Graph, Symplectic Group, Generalized Symplectic Graph.
Mathematics Subject Classification [2010]: Primary: 05E18; Secondary: 05C25

1 Introduction

In this paper, a graph $\Gamma = \Gamma(V, E)$ is considered as a simple undirected graph with vertex-set $V(\Gamma) = V$, and edge-set $E(\Gamma) = E$.

Let $\Gamma(V, E)$ and $\Lambda(V', E')$ be two graphs. The mapping $\beta : V \longrightarrow V'$ is a homomorphism from Γ to Λ if $v, w \in V(\Gamma)$ are adjacent in Γ, then $\beta(v), \beta(w) \in V'(\Lambda)$ are adjacent in Λ. An isomorphism between Γ and Λ is a bijection homomorphism $\beta : V \longleftrightarrow V'$ with $v, w \in V(\Gamma)$ are adjacent in Γ, if and only if $\beta(v), \beta(w) \in V'(\Lambda)$ are adjacent in Λ.
An automorphism of a graph Γ is an isomorphism from Γ to itself. The set of all automorphisms of Γ, with composition of functions, is called the automorphism group of Γ and denoted by $Aut(\Gamma)$.

In most situations, it is difficult to determine the automorphism group of a graph, but there are various in the literature and some of the recent works come in the references [3, 5]. Now, let Γ be a graph with automorphism group $G = Aut(\Gamma)$. For vertex $v \in V(\Gamma)$, let G_v denote the stabilizer subgroup of vertex v; that is, the subgroup of G containing of those automorphism that fix v. From first isomorphism theorem, we know that:

$$[G : G_v] = \frac{|G|}{|G_v|} \leq |V(\Gamma)|.$$

The graph Γ is called vertex-transitive if $G = Aut(\Gamma)$ acts transitively on $V = V(\Gamma)$. In other words, for any two vertices $v, w \in V(\Gamma)$ there is an automorphism $\alpha \in Aut(\Gamma)$ such that $\alpha(v) = w$. Now if Γ is a vertex-transitive graph, then for each vertex $v \in V(\Gamma)$, we have

$$\frac{|G|}{|G_v|} = |V| \Longrightarrow |G| = |G_v||V|.$$

[1]speaker

Let $\Gamma = (V, E)$ be a graph. The action of $Aut(\Gamma)$ on $V(\Gamma)$ induces an action on $E(\Gamma)$, by the rule $\beta\{x, y\} = \{\beta(x), \beta(y)\}$, where $\beta \in Aut(\Gamma)$, and $\{x, y\} \in E(\Gamma)$. Γ is called edge transitive if this action is transitive.

In this paper, let R be a commutative ring with identity element 1, and let V be a free R - module of R - dimension $n \geq 2$. The symplectic form β is a bilinear form $\beta : V \times V \longrightarrow R$, such that $\beta(x, x) = 0$ for all $x \in V$. The pair (V, β) is called a symplectic space. The symplectic form $\beta : V \times V \longrightarrow R$ is called nonsingular, when the R-module homomorphism from V to $V^* = Hom_R(V, R)$ given by $x \longmapsto \beta(\ , x)$ is an isomorphism, for all $x \in V$. In the sequence, assume that β is a nonsingular symplectic form.

Recall that an element x in V is unimodular if there is an $f \in V^*$ such that $f(x) = 1$. For $x \in V$, we call Rx a line. A hyperbolic pair$\{x, y\}$ is a pair of unimodular vectors in V with the property that $\beta(x, y) = 1$. The module $H = Rx \bigoplus Ry$ is called a hyperbolic plane.

Suppose that (V, β) and (V', β') are two symplectic spaces. An isometry from (V, β) to (V', β') is an R - isomorphism $\sigma : V \longrightarrow V'$ such that:

$$\beta(x_1, x_2) = \beta'(\sigma(x_1), \sigma(x_2)) \text{ for every elements } x_1, x_2 \in V.$$

It is easy to verify that the set of all isometries from (V, β) to (V, β) is a group; this group is called symplectic group over V and denoted by $SP_R(V)$.

Definition 1.1. Let $B = \{v_1, \ldots, v_{2n}\}$ be a basis for the symplectic space (V, β). The matrix $B = (b_{ij})_{1 \leq i,j \leq 2n}$, where $b_{ij} = \beta(v_i, v_j)$ is called the matrix of the form β over B.

The following theorem has been obtained from the definition of symplectic space and has an easy proof.

Theorem 1.2. *Let (V, β) and (W, β') be two symplectic spaces with $dimV = dimW = 2n$. Suppose that B_1 and B_2 are ordered basis of V and W respectively. If we denote the matrices of β and β' with respects to the above basis by B and C respectively, then $T : V \longrightarrow W$ is an isometry from V to W if and only if $A^tCA = B$, where A is matrix of T with respect to B_1.*

Corollary 1.3. *Let R be a stably free ring and (V, β) be symplectic space over R. Then*

$$SP_R(V) = \{A | A \textit{ is invertable and } A^tJA = J\}$$

where J is blockdiagonal matrix as follow:

$$J = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

In [2] it is proved that, $Z(SP_R(V)) = \{\pm I_{2n}\}$, where $Z(SP_R(V))$ denotes the center of the group $SP_R(V)$. A commutative ring R have a stable range one if for all $\alpha, \beta \in R$ with $\langle \alpha, \beta \rangle = R$, there exist a δ in R such that $\alpha + \delta\beta \in R^*$.

Lemma 1.4. *[2] Let R be a commutative ring with stable range 1 and $2 \in R$ be an unit. Let V be a symplectic space over R. Then $SP_R(V)$ acts transitively on unimodular vectors and on hyperbolic planes.*

Definition 1.5. Generalized symplectic group over ring R is denoted by $GSP_R(V)$ and defined as follow:

$$GSP_R(V) = \{T | T \text{ is invertible over } R \text{ and } TJT^t = kJ \text{ for some } k \in R^*\}.$$

2 Main Results

For all terminologies and notations not defined here, we follow [1, 2]. We now define a class of regular graphs, which is known as symplectic graphs.

Definition 2.1. Let (V, β) be a symplectic space over ring R. The symplectic graph over $SP_R(V)$ denoted by $\mathcal{G}SP_R(V)$, is a graph with vertex- set

$$\{Rx | x \text{ is unimodular in } V\},$$

and two vertices Rx and Ry are adjacent if and only if $\beta(x, y) \in R^*$.

This adjacency is well defined, since if x_1, x_2, y_1, y_2 are unimodular elements in V with $Rx_1 = Rx_2$ and $Ry_1 = Ry_2$, then there exist $\lambda, \mu \in R^*$ such that $x_1 = \lambda x_2$ and $y_1 = \mu y_2$. Therefore

$$\beta(x_1, y_1) \in R^* \iff \beta(\lambda x_2, \mu y_2) \in R^*$$
$$\iff \lambda\mu\beta(x_2, y_2) \in R^* \iff \beta(x_2, y_2) \in R^*.$$

Now from lemma 1.4 we have the following lemma that proved in [2].

Lemma 2.2. *Let R be a commutative ring with stable range 1 and $2 \in R$ be an unit. Then the symplectic graph $\mathcal{G}SP_R(V)$ is vertex-transitive and edge-transitive.*

We now define a symplectic graph over $R = \mathbb{Z}_{p^n}$. Let $V^{2v} \subseteq \mathbb{Z}_{p^n}^{(2v)}$ be a set of elements $(a_1, a_2, \dots, a_{2v})$, where for all $1 \leq i \leq 2v$, $a_i \in \mathbb{Z}_{p^n}$ and there is an $i \in \{1, \dots, 2v\}$ such that a_i is invertible in $\mathbb{Z}_{p^n}$. We define an equivalence relation $\sim_{p^n}$ on V by the following rule:

$$(a_1, a_2, \dots, a_{2v}) \sim_{p^n} (b_1, b_2, \dots, b_{2v}) \iff (a_1, a_2, \dots, a_{2v}) = \lambda(b_1, b_2, \dots, b_{2v}),$$

for some $\lambda \in \mathbb{Z}_{p^n}^*$.
Let $[a_1, \dots, a_{2v}]$ denotes the equivalence class of $(a_1, \dots, a_{2v})$ with respect to $\sim_{p^n}$, and let $V_{\sim_{p^n}}^{(2v)}$ be the set of all equivalence classes. We define the bilinear form $\beta : V_{\sim_{p^n}}^{(2v)} \times V_{\sim_{p^n}}^{(2v)} \longrightarrow R$ by the rule $\beta(x, y) = xJy^t$.
The symplectic graph module p^n on $\mathbb{Z}_{P^n}^{(2v)}$, relative to J which is denoted by $SP_{p^n}^{(2v)}$, is a graph with vertex-set $\{[a_1, \dots, a_{2v}] | (a_1, \dots, a_{2v}) \in V^{(2v)}\}$ and adjacency defined by

$$[a_1, \dots, a_{2v}] \text{ adjacent to } [b_1, \dots, b_{2v}] \text{ if and only if } \beta(x, y) \in \mathbb{Z}_{P^n}^*,$$

where $x = (a_1, \dots, a_{2v})$ and $y = (b_1, \dots, b_{2v})$. In [4], it is proves that $SP_{p^n}^{(2v)}$ is a vertex and edge-transitive graph.
In the first step, note that β is a symplectic form over $\mathbb{Z}_{P^n}^{(2v)}$.

Lemma 2.3. *Each element of $V := V_{\sim_{p^n}}^{(2v)}$ is unimodular.*

By previous lemma, we conclude that for $R = \mathbb{Z}_{P^n}$, $\mathcal{G}SP_R(v)$ is isomorphic to $SP_{P^n}^{(2v)}$.
In [2], it is proved that $\mathbb{Z}_{P^n}$ has a stable range one, and we know that for $p \geq 2$, 2 is unit in $\mathbb{Z}_{P^n}$, where p is prime. Then by lemma 2.2. we conclude that $SP_{P^n}^{(2v)}$ is vertex-transitive and edge-transitive.

Lemma 2.4. *Let p be a prime integer and $R = \mathbb{Z}_{P^n}$ and $V = \mathbb{Z}_{P^n}^{(2v)}$. Suppose that $T \in GSP_R(V)$. We define $\sigma_T : V \longrightarrow V$ by the rule $\sigma_T(x) = R(xT)$ for all unimodular elements $x \in V$. Then $T \in GSP_R(V)$ if and only if $\sigma_T \in Aut(\mathcal{G}SP_R(V))$.*

We now proceed to proving the main result of this paper.

Theorem 2.5. *Let $R = \mathbb{Z}_{P^n}$ and $V = \mathbb{Z}_{P^n}^{(2v)}$, then*

$$Aut(\mathcal{G}SP_R(V)) = \frac{GSP_R(V)}{kI},$$

for some $k \in R^*$.

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Posters

Non-isomorphic Regular Graphs With Less Than 16 Vertices

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Abstract

In this paper, we find an innovative method for non-isomorphic regular graphs with less than 16 vertices. In other words, we show that we can create several distinct shapes that are still regular graphs. This simulation works in a way that the first system, although not similar in appearance, but in terms of system relations and logic, achieves the same goal of the first system, so that the r-regular graphs of order n convert to r′-regular graphs of order of n′ that does not make a difference in the essence of the theorem. In other words, we can turn complex geometric shapes into simple routine shapes.

Keywords: Regular graph
Mathematics Subject Classification [2010]: Primary:05C70; Secondary: 05C30

1 Introduction

The term graph has only recently appeared in the mathematical literature although it dates back to the time of the Swiss mathematician Leonardo Euler (1707-1783). However, a strong and continuing interest in graph theory, as a branch of mathematics, became apparent from the 1930s onwards, and today this theory is one of the most prolific and popular branches of mathematics and computer science because of its applicability in a vast variety of the issues of modern society. Once a problem is formulated in graph language, it will be much easier to understand. Nowadays, graph theory is one of the most important topics in the field of discrete mathematics. Graphs are, in fact, the mathematical models for a discrete set, whose members are interconnected in some way. The members of this set may be humans or relatives, or friends, and so on. The members of this set can be the junctions of the wires of a power grid and their relationship, the wires between two points, or the elements of the set can be the atoms of a molecule and their connections, the chemical connections. Graph theory is also rooted in games and puzzles, but today it is widely used not only in mathematics but also in other sciences such as economics, psychology, genetics and archeology.

2 Non-isomorphic regular graphs

Definition 2.1. Two graphs which contain the same number of graph vertices connected in the same way are said to be isomorphic. Formally, two graphs G and H with graph vertices $V_n = \{1, 2, ..., n\}$ are said to be isomorphic if there is a permutation p of V_n such that $\{u, v\}$ is in the set of graph edges $E(G)$ iff $\{p(u), p(v)\}$ is in the set of graph edges $E(H)$.

Definition 2.2. A graph is said to be a regular graph if all the vertices are of the same degree, or in other words, an equal number of edges pass through all the vertices.

Let G and H be two regular graphs. Obviously, if there is not any permutation of their vertices which satisfies in conditions of definition 2.1, we say G and H are **non-isomorphic regular graphs**.

[1]speaker

3 Innovative formula for the r-regular graph of order n

Definition 3.1. An integer a is said to be divisible by an integer $b \neq 0$ whenever there exists an integer q such that $a = bq$. Then we write $b \mid a$ and say a is divisible by b.

The following formula are valid for numbers 1 to 15. For two-digit numbers n except for $r = 2$, (it is also true for one-digit numbers n with $r = 2$). In [2], it is proved that 2-regular graphs are union of cycles. Let n be the number of vertices of a graph. If r and q are degree of vertex and the number of edges respectively, then, the Necessary and sufficient condition for using this formula is $nr = 2q$ which is the same condition of regular graphs.
Number of separate nodes:

1. $n - (r + 2), (r + 2)$
2. $n - (r + 2), (r + 2)$ divided by their common multiple (the resulted number means that the same number can be plotted). At the end of the discussion, we will address this point for a better understanding.
3. itself

Conditions:

1. $\{q \mid q \in N\}$
2. $n - (r + 2) > r$

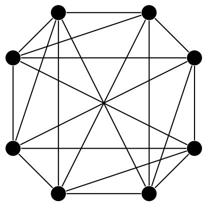
According to the data that the formula gives us, we can easily find the formula for all the numbers by analysis. Since the formula for 2-regular graphs has been already found and recorded, these two formulae can be easily considered as the complement of each other and consist all the numbers.
The results obtained with the formula for r-regular graphs of order n are expressed by some examples as the following:

Example 3.2. 2-regular graph of order 6
6-(3+2) , (3+2) O
6-(2+2) , (2+2) O
6-(2+1) , (2+1) P
+itself

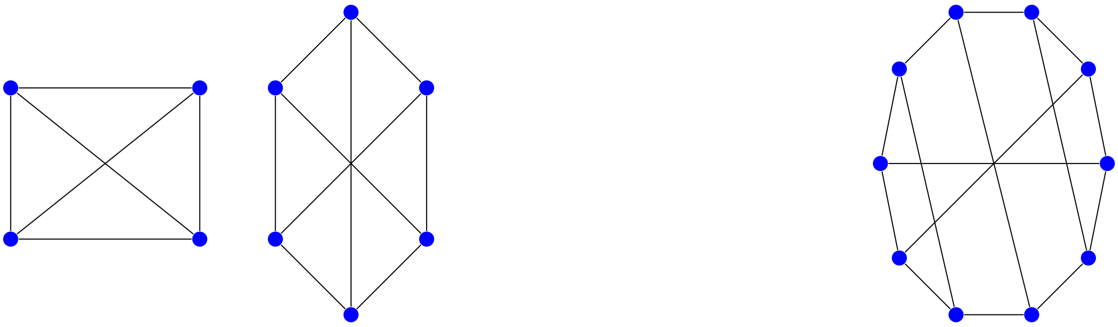
Example 3.3. 2-regular graph of order 7
7-(3+2) ,(3+2) O
7-(2+2) , (2+2) P
7-(1+2) , (1+2) O (repetitive)
+itself

Example 3.4. 3-regular graph of order 8
8-(4+2) , (4+2) O
8-(3+2) , (3+2) O
8-(2+2) , (2+2) P
8-(1+2) , (1+2) O
+itself

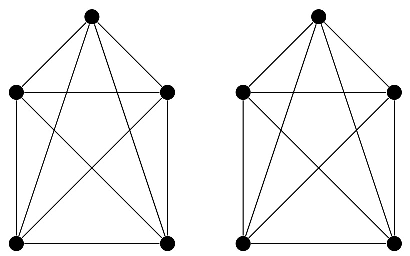
Example 3.5. 5-regular graph of order 8
8-(6+2) , (6+2) O
8-(5+2) , (5+2) O
8-(4+2) , (4+2) O
8-(3+2) , (3+2) O
8-(2+2) , (2+2) O
8-(1+2) , (1+2) O
+itself

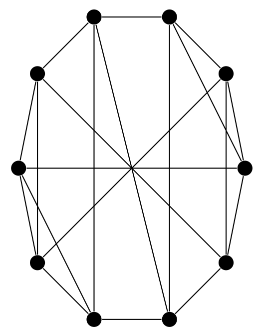


Example 3.6. 3-regular graph of order 10
10-(4+2) , (4+2) O
10-(3+2) , (3+2) O
10-(2+2) , (2+2) P
10-(1+2) , (1+2) O
+itself



Example 3.7. 4-regular graph of order 10
10-(5+2) , (5+2) O
10-(4+2) , (4+2) O
10-(3+2) , (3+2) P
10-(2+2) , (2+2) O
10-(1+2) , (1+2) O
+itself





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Some Properties of Sum of the degrees of all neighbors of a vertex in fuzzy graph

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Abstract

Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $u \in V$. We define $S_G^F(u) = \sum_{vu \in \mathcal{E}} \mu(vu) \deg_G v$. In this paper, we present some properties of $S_G^F(u)$ on fuzzy graphs and establish relations between the Zagreb indices and sum of the degrees of all neighbors of a vertex.

Keywords: fuzzy graph, fuzzy common neighborhood graph, sum of the degrees of all neighbors of a vertex.
Mathematics Subject Classification [2010]: Primary: 05C07; Secondary: 03E72

1 Introduction

The concept of graph theory was first introduced by Euler. In 1965, L. A. Zadeh discussed the fuzzy set [11]. The first definition of a fuzzy graph was given by Kaufmann, which was based on Zadeh's fuzzy relations in [1]. However, the development of fuzzy graph theory is due to the ground setting papers of Rosenfeld [2] and Yeh and Bang [12]. In Rosenfeld's paper, basic structural and connectivity concepts were presented while Yeh and Bang introduced different connectivity parameters and discussed their application. Rosenfeld obtained the fuzzy analogs of several graph-theoretic concepts like bridges, paths, cycles, trees, and connectedness. Most of the theoretical development of fuzzy graph theory is based on Rosenfeld's initial work. J. N. Mordeson studied fuzzy line graphs and developed its basic properties, in 1993 [8].

The main purpose of this paper is to define some concepts in fuzzy graphs and get results about them that are also true in ordinary graphs. In this section, we provide formal definitions, basic concepts, and properties of fuzzy graphs. For simplicity, we consider only undirected fuzzy graphs, unless otherwise specified. Thus, the edges of the fuzzy graph are unordered pairs of vertices. First, we go through some basic definitions from [8, 13].

Definition 1.1. A fuzzy subset of a non-empty set S is a map $\sigma : S \to [0, 1]$ which assigns to each element x in S a degree of membership $\sigma(x)$ in $[0, 1]$ such that $0 \leq \sigma(x) \leq 1$.

If S represents a set, a fuzzy relation μ on S is a fuzzy subset of $S \times S$. In symbols, $\mu : S \times S \to [0, 1]$ such that $0 \leq \mu(x, y) \leq 1$ for all $(x, y) \in S \times S$.

Definition 1.2. Let σ be a fuzzy subset of a set S and μ a fuzzy relation on S. Then μ is called a fuzzy relation on σ if $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in S$ where $\wedge$ denote minimum.

[1]speaker

Let V be a nonempty set. Define the relation $\sim$ on $V \times V$ by for all $(x, y), (u, v) \in V \times V$, $(x, y) \sim (u, v)$ if and only if $x = u$ and $y = v$ or $x = v$ and $y = u$. Then it is easily shown that $\sim$ is an equivalence relation on $V \times V$. For all $x, y \in V$, let $[(x, y)]$ denote the equivalence class of (x, y) with respect to $\sim$. Then $[(x, y)] = \{(x, y), (y, x)\}$. Let $\mathcal{E}_V = \{[(x, y)] | x, y \in V, x \neq y\}$. For simplicity, we often write $\mathcal{E}$ for $\mathcal{E}_V$ when V is understood. Let $E \subseteq \mathcal{E}$. A graph is a pair (V, E). The elements of V are thought of as vertices of the graph and the elements of E as the edges. For $x, y \in V$, we let xy denote $[(x, y)]$. Then clearly $xy = yx$. We note that graph (V, E) has no loops or parallel edges.

Definition 1.3. A fuzzy graph $G = (V, \sigma_G, \mu_G)$ is a triple consisting of a nonempty set V together with a pair of functions $\sigma := \sigma_G : V \to [0, 1]$ and $\mu := \mu_G : \mathcal{E} \to [0, 1]$ such that for all $x, y \in V$, $\mu(xy) \leq \sigma(x) \wedge \sigma(y)$.

The fuzzy set σ is called the fuzzy vertex set of G and μ the fuzzy edge set of G. Clearly, μ is a fuzzy relation on σ.

Definition 1.4. A path P in a fuzzy graph $G = (V, \sigma, \mu)$ is a sequence of distinct vertices $x_0, x_1, \cdots, x_n$ (except possibly x_0 and x_n) such that $\mu(x_{i-1}x_i) > 0$ for $i = 1, \cdots, n$. Here n is called the length of the path. We call P a cycle if $x_0 = x_n$ and $n \geq 3$. Two vertices that are joined by a path are called connected.

Definition 1.5. Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The degree $x \in V$ is denoted by $\deg_G(x)$ and defined as $\deg_G(x) = \sum_{y \in V} \mu(xy)$.

Definition 1.6. Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The size of G is denoted by $S(G)$ and defined as $\sum_{xy \in \mathcal{E}} \mu(xy)$.

Definition 1.7. The *fuzzy common neighborhood graph* or briefly *fuzzy congraph* of $G = (V, \sigma, \mu)$ is a fuzzy graph as $con(G) = (V, \omega, \lambda)$ such that $\omega(x) = \sigma(x)$ and $\lambda(uv) = min_{x \in H}\{\mu(ux).\mu(vx)\}$, where $H = N_G(u) \cap N_G(v)$.

Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $V = \{v_1, v_2, ..., v_p\}$, $\mathcal{E} = \{e_1, e_2, ..., e_q\}$ the vertex set and the edge set of G, respectively.

The *adjacency matrix* of fuzzy graph G is the $p \times p$ matrix $A_F = A_F(G)$ whose (i, j) entry denoted by a_{ij}, is defined by $a_{ij} = \mu(v_i v_j)$.

Definition 1.8. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrix of size $m \times n$. Then we define $C = A \odot B$ is the $m \times n$ matrix whose (i, j) entry denoted by $a_{ij} \times b_{ij}$.

Two old and much studied degree–based graph invariants are the so-called *first and second Zagreb indices*, defined as

$$M_1(G) = \sum_{v \in V(G)} \deg(v)^2 \qquad \text{and} \qquad M_2(G) = \sum_{uv \in E(G)} \deg(u)\,\deg(v)\ .$$

Also, in [14], the authors defined two new indices $N_1(G)$ and $N_2(G)$ as follows.

$$\begin{aligned} N_1(G) &= \sum_{u \in V(G)} \deg_G(u)\,\deg_{con(G)}(u) \text{ and} \\ N_2(G) &= \sum_{uv \in \varepsilon(con(G))} \deg_G(u)\,\deg_G(v). \end{aligned}$$

For details on their history, mathematical properties and chemical applications see [5, 3, 6, 10, 7, 4] and the references cited therein. Now, we define these indices in a fuzzy graph.

Definition 1.9. Let $G = (V, \sigma, \mu)$ be a fuzzy graph.

(1) $M_1^F(G) := \sum_{v_i \in V} \deg_G^2(v_i)$;

(2) $M_2^F(G) := \sum\limits_{v_iv_j \in \mathcal{E}} \mu(v_iv_j) \deg_G(v_i) \deg_G(v_j)$;

(3) $F^F(G) := \sum\limits_{v_i \in V} \deg_G^3(v_i)$;

(4) $N_1^F(G) := \sum\limits_{u \in V(G)} \deg_G(u)\ \deg_{con(G)}(u)$;

(5) $N_2^F(G) := \sum\limits_{uv \in \varepsilon(con(G))} \mu(uv) \deg_G(u)\ \deg_G(v)$.

Definition 1.10. Let $A = [a_{ij}]_{m \times n}$. Then, we define

$$S(A) = \sum_{1 \leq i \leq m,\ 1 \leq j \leq n} a_{ij}.$$

2 Main Results

In this section, first, we define some fuzzy graph operations that were first studied in [9] by Mordeson and Peng in 1994. Also, we define sum of the degrees of all neighbors of a vertex in a fuzzy graph and then we investigate some properties of it. We will establish relations between the fuzzy Zagreb indices and sum of the degrees of all neighbors of a vertex.

Definition 2.1. Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs such that $V_1 \cap V_2 = \varnothing$. Union of two fuzzy graphs G_1 and G_2 is denoted by $G_1 \cup G_2 = (V, \sigma, \mu)$ such that $V = V_1 \cup V_2$,

$$\sigma(v) = \begin{cases} \sigma_1(v) & , \quad v \in V_1 \\ \\ \sigma_2(v) & , \quad v \in V_2 \end{cases} \text{ and } \mu(uv) = \begin{cases} \mu_1(uv) & , \quad u, v \in V_1 \\ \mu_2(uv) & , \quad u, v \in V_2 \\ 0 & , \quad o.w \end{cases}.$$

It is easy to see $G_1 \cup G_2$ is a fuzzy graph.

Definition 2.2. Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs such that $V_1 \cap V_2 = \phi$. Sum of two fuzzy graphs G_1 and G_2 is denoted by $G_1 \vee G_2 = (V, \sigma, \mu)$ such that $V = V_1 \cup V_2$,

$$\sigma(v) = \begin{cases} \sigma_1(v) & , \quad v \in V_1 \\ \\ \sigma_2(v) & , \quad v \in V_2 \end{cases}, \ \mu(uv) = \begin{cases} \mu_1(uv) & , \quad u, v \in V_1 \\ \mu_2(uv) & , \quad u, v \in V_2 \\ k & , \quad u \in V_1,\ v \in V_2 \end{cases}$$

, where $k = \min\{\sigma_1(u), \sigma_2(v)\}$ for every $u \in V_1$ and $v \in V_2$.

Definition 2.3. Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs. The Cartesian product of graphs G_1 and G_2 is denoted by $G_1 \times G_2 = (V, \sigma, \mu)$ is a fuzzy graph such that $V = V_1 \times V_2$,
$\sigma((u, v)) = \sigma_1(u) \vee \sigma_2(v)$, where $\vee$ is denoted maximum and

$$\mu((u, v)(u', v')) = \begin{cases} \mu_2(vv') & , \quad \text{if } u = u' \\ \mu_1(uu') & , \quad \text{if } v = v' \\ 0 & , \quad o.w \end{cases}.$$

It is easy to show that $d_{G_1 \times G_2}((u, v)) = \deg_{G_1}(u) + \deg_{G_2}(v)$.

Let $G = (V, \sigma, \mu)$ be a fuzzy graph the neighbor of vertex v is denoted by $N_G(v)$ and is defined as follows.

$$N_G(v) = \{u \in V \mid \mu(uv) > 0\}.$$

Let G be a graph. Sum of the degrees of all neighbors of vertex u in G denoted by $S_G(u)$ and define as $S_G(u) = \sum\limits_{vu \in \mathcal{E}} \deg_G v$. Now, we will extend it to a fuzzy graph.

Definition 2.4. Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $u \in V$. We define $S_G^F(u) = \sum\limits_{vu \in \mathcal{E}} \mu(vu) \deg_G(v)$ (v is a neighbor of vertex u).

Theorem 2.5. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then*

$$\sum_{u \in V} S_G^F(u) = M_1^F(G).$$

Theorem 2.6. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then*

$$\sum_{u \in V} \deg_G(u) S_G^F(u) = 2M_2^F(G).$$

Theorem 2.7. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $con(G) = (V, \omega, \lambda)$ the fuzzy congraph of G. If G has no cycles of size 4, then*

$$d_{con(G)}(v) = S_G^F(u) - \sum_{u \in V} \mu^2(uv), \qquad v \in V.$$

Theorem 2.8. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph and A_F be the adjacency of G. If G has no cycles of size 4, then*

$$N_1^F(G) + S(A_F.(A_F \odot A_F)) = 2M_2^F(G).$$

Since in the ordinary graph $A \odot A = A$, by the above theorem, and by [14, Lemma 2.3(2)], we deduce the following result.

Corollary 2.9. *Let G be a graph which has no cycles of size 4. Then,*

$$M_2(G) = \frac{1}{2}(N_1(G) + M_1(G)).$$

Theorem 2.10. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph such that it has no cycles of size 4, then*

$$2S(con(G)) = \frac{1}{2} M_1^F(G) - S(A_F \odot A_F).$$

From the above theorem, and by [14, Lemma 2.3(1)], we deduce the following result in a graph.

Corollary 2.11. *Let G be a (p, q)-graph and have no cycles of size 4. Also, let $con(G)$ be a (p, q')-graph. Then, $q' = \frac{1}{2} M_1(G) - q$.*

Lemma 2.12. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph such that G is not a null fuzzy graph. Then*

$$\frac{\delta}{2} \leq \frac{M_2^F(G)}{M_1^F(G)} \leq \frac{\Delta}{2}.$$

Theorem 2.13. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then*

$$\sum_{uv \in \mathcal{E}} \mu(uv)[S_G^F(u) + S_G^F(v)] = 2M_2^F(G).$$

Let $S_G(u)$ be sum of the degrees of all neighbors of vertex u in graph G. The following result can be obtained from the above theorem.

Corollary 2.14. *Let G be a graph having no cycles of size 4. Then, $\sum_{uv \in E} (S_G(u) + S_G(v)) = 2M_2(G)$.*

Lemma 2.15. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then, for every $v \in V$ the following holds:*
$\sum_{v_i v_j \in E} \mu(v_i v_j)(\deg_G^k(v_i) + \deg_G^k(v_j)) = \sum_{v_i \in V} \deg_G^{k+1}(v_i)$.

In particular,

$$\sum_{v_i v_j \in E} \mu(v_i v_j)(\deg_G(v_i) + \deg_G(v_j)) = \sum_{v_i \in V} \deg_G^2(v_i) = M_1^F(G),$$

and

$$\sum_{v_i v_j \in E} \mu(v_i v_j)(\deg_G^2(v_i) + \deg_G^2(v_j)) = \sum_{v_i \in V} \deg_G^3(v_i) = F^F(G).$$

Theorem 2.16. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph such that it has no cycles of size 4, then*

$$\sum_{uv \in \mathcal{E}} (\deg_G(u) S_G^F(v) + \deg_G(v) S_G^F(u)) = F^F(G) + 2N_2^F(G).$$

From the above theorem, we deduce the following result.

Corollary 2.17. *Let G be a graph having no cycles of size 4. Then,*

$$\sum_{uv \in E} (\deg_G(u) S_G(v) + \deg_G(v) S_G(u)) = F + 2N_2(G),$$

where $F = \sum_{v \in V} \deg(v)^3$.

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Algorithms for Vertex Cover Problem

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Abstract

A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. The Minimum Vertex Cover (MinVC) problem is to find the minimum sized vertex cover in a graph. MinVC is a prominent combinatorial optimization problem with many applications, such as network security. Cook introduced the problem and in later, he showed that the vertex cover problem is NP-Hard. Karp showed that finding the solution of minimal vertex cover problem in a graph is an NP-complete problem. Furthermore, it is NP-hard to approximate MVC within any factor smaller than 1.3606. In this work, we introduce study this problem and developed a new algorithm, called NOVC, for vertex cover problem that its complexity is nc/δ. where n is the number of vertices, c is the number of vertices in the vertex cover and δ is the minimum degree of the graph. The running time is better than the algorithms being compared to.

Keywords: Vertex cover problem, Local search algorithms, Heuristic algorithms,Vertex coloring problem, Maximum clique problem, Approximation algorithms.
Mathematics Subject Classification [2010]: Primary: 22D15, 43A10; Secondary: 43A20, 46H25

1 Introduction

A vertex cover of graph is a set of vertices such that each rdge of the graph is incident to at least one vertex of the set. The mimimum vertex cover problem is to find the minimum sized vertex cover in a graph.minimum vertex cover is a prominent combinatorial optimization problem with many applications, such as network security. The issue of vertex cover is one of the compound optimization problem, which was first raised by Cook in 1968 and in 1971 Cook proved that this was difficult in the NP-hard category. In 1972, Richard Carp proved that decision of vertex cover problem is the NP-complete.

1.1 MDG Algorithm

The idea of the algorithm is that each time it tries to cover the largest number of edges by selection the vertex with the largest degree and And removes the edges adjacent to the vertex with the largest degree and updates the degree of adjacent vertices. If there are several vertices with the largest degree, the algorithm selects one of them randomly and This continues until the edge set is empty. The implementation of the algorithm on an example is shown in figure 1and the minimum vertex cover is {1,2,4,6}.

MDG algorithm
Input G(V,E)
S $\longleftarrow$ E, MVC $\longleftarrow \phi$

[1]speaker

3: While S $\neq \phi$ do
4: pick any u,v $\in$ S
5: MVC $\longleftarrow$ MVC $\bigcup$ u,v
6: delete all edges incident to either u or v from S
7: end While

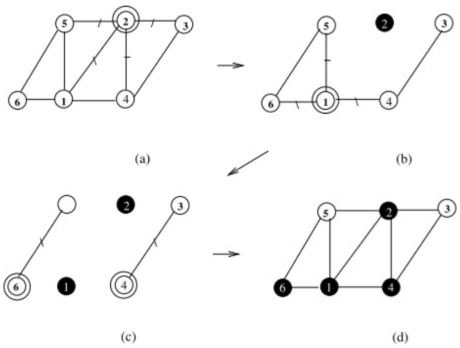

Figure 1: Execution of MDG algorithm

1.2 AVD Algorithm

The idea of the algorithm is that it first considers the set of all vertices of a given graph as the set of vertex cover And until the size of the vertex cover set reaches K Each time it removes the vertex from the vertex cover set according to the following two procedures so that the remaining sets is vertex cover.

AVD Algorithm:
Part I.
1: For i=1, 2, ..., n in turn
2: Initialize the vertex cover C_i=V-i
3: Perform procedure1 on C_i
4: For r=1,2,..., n-k perform procedure2 repeated r times.
5: The result is a minimal vertex cover C_i
Part II. For each pair of minimal vertex cover C_i,C_j found in Part I
6Initialize the vertex cover $C_{i,j} = C_i \cup C_j$
7: Perform procedure1 on $C_{i,j}$
For r=1, 2, ..., n-k perform procedure2 repeated r time.

2 Main Results

we proposed a new algorithm to solve a vertex cover problem and called NOVA This algorithm is in many cases faster and more accurate than other algorithms. The idea is to always add the vertices of the vertex with a minimum degree to the vertex cover set. When there are several vertices with a minimum degree, it considers the vertex whose sum of the degrees of its neighboring vertices is maximum. The implementation of the algorithm on an example is shown in figure 1 and the minimum vertex cover is {b,d,g,k,h}.

Algorithm: NOVA
δ : vertex with minimum degree, D_{V_i} : degree of vertex i , $\varDelta$: vertex with maximum degree
δ: vertex with minimum degree
S_{min}: for each vertex having the minimum sum of degrees of its neighbors
S_{max}: for each vertex having the minimum sum of degrees of its neighbors
t: for every vertex its counter of neighbors are inside in vertex cover set
Input: G = (V, E)
Output: VC that is a set of vertices that form a vertex cover.
1: Let V C = ϕ

while E $\neq \phi$ do
    find a vertex v with minimum degree (δ)
if $D_{V_i} = \delta$ and $D_{V_j} = \delta$ for some vertex i and j, then select a vertex with S_{max}
    delete v from the graph together with all edges incident to it.
    add vertices of N(v) to VC.
    increase t_i by one unit for all neighbors of vertices in N(v).
Return C;

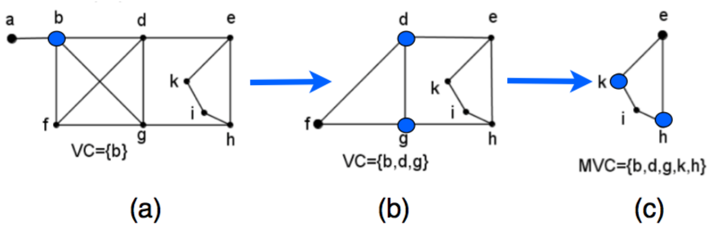

Figure 2: Execution of NOVA algorithm

2.1 performance comparisons to other algorithm

To test the performance f this algorithm we ran algorithm on several graph with 50,100, 200, 500, 700 vertex and different edge with 100 Repetition. The percentage of success of an algorithm in obtaining the optimal answer is equal to the ratio of the number of examples for which the algorithm has been able to obtain the minimum coverage to the total number of examples on which the algorithm has been executed. In figure 3 we show the success rate of these 3 algorithms.

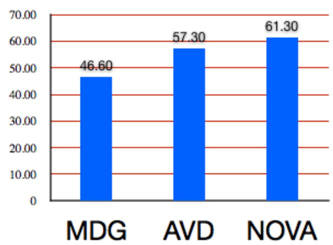

Figure 3: Success rate of algorithms

The error percentage is equal to the ratio of the obtained answer to the optimal answer. The error rate of this algorithm is shown in the average and worst case in figure 4

According to the Charts the algorithm NOVA has a better success rate and a lower error rate than other algorithms.

Theorem 2.1. *The temporal complexity of graph algorithm with n vertex is* $O(n^2 log(n))$

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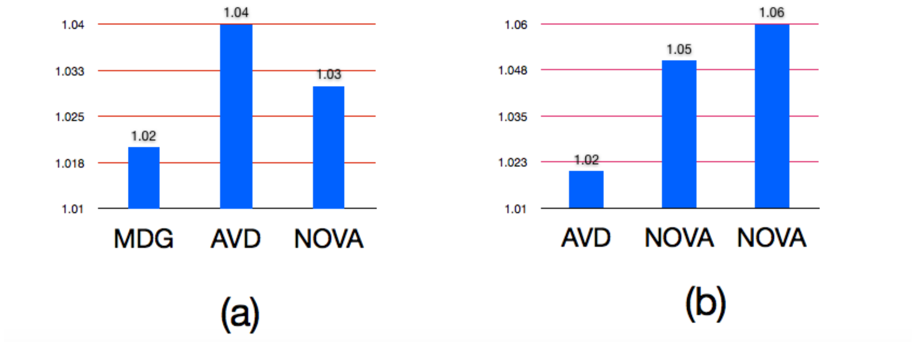

Figure 4: Error rate of algorithms. (a) average case approximation, (b) worest case approximation

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Zero divisor graph in $MV-$algebras

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Abstract

In this article, the relationship between minimal prime ideals and zero divisors are investigated. We prove the set of all zero divisors $MV-$algebra contains at least one prime ideal of $MV-$algebra and the set of all zero divisors of $MV-$algebra is a union of all minimal prime ideals of $MV-$algebra. Also, it is proved that every element of a minimal prime ideal of $MV-$algebra is a zero divisor. We introduce zero divisor graph in $MV-$algebra and prove that this graph is connected.

Keywords: zero divisor graph, minimal prime, $MV-$algrebra.
Mathematics Subject Classification [2010]: Primary: 06D35; Secondary: 06B10; Thirdly: 97K30

1 Introduction and Preliminaries

C.C. Chang introduced MV-algebras as algebraic models for ukasiewicz logic to give its algebraic analysis and proved completeness of ukasiewicz logic with respect to the variety of all MV-algebras. [1]. F. Forouzesh et al. defined the set of all zero divisors of an $MV-$ algebra A and investigated some properties of them[3]. We investigate the relationship between minimal prime ideals and zero divisors. It is proved that every element of a minimal prime ideal of $MV-$algebra is a zero divisor. We demonstrated with an example that the opposite is not necessary true. We recollect some definitions and results which will be used in the sequed:

Definition 1.1. [1] An $MV-$algebra is a structure $(A, \oplus, ^*, 0)$ where $\oplus$ is a binary operation, *, is a unary operation, and 0 is a constant such that the following axioms are satisfied for any $x, y \in A$:
$(MV1)$ $(A, \oplus, 0)$ is an abelian monoid; $(MV2)$ $(x^*)^* = x$;
$(MV3)$ $0^* \oplus x = 0^*$; $(MV4)$ $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Note that we have $1 = 0^*$ and the auxiliary operation $\odot$ which are as $x \odot y = (x^* \oplus y^*)^*$.
We recall that the natural order determines a bounded distributive lattice structure such that

$$x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*) \quad \text{and} \quad x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

Also for any two elements $x, y \in A$, $x \leq y$ if and only if $x^* \oplus y = 1$ if and only if $x \odot y^* = 0$. In this paper, A is an $MV-$algebra.

Definition 1.2. [2] An ideal of A is a nonempty subset I of A satisfying the following conditions:
$(I1)$ If $x \in I,\ y \in A$ and $y \leq x$, then $y \in I$, $(I2)$ If $x, y \in I$, then $x \oplus y \in I$.
We denote by $Id(A)$ the set of all ideals of A.

[1] speaker

Definition 1.3. [2] Let I be an ideal of A. If $I \neq A$, then I is a proper ideal of A.
• [2] A proper ideal I of A is called prime ideal if for all $x, y \in A, x \wedge y \in I$, then $x \in I$ or $y \in I$.
We denote by $Spec(A)$ the set of all prime ideals of an $MV-$algebra A.
• [2] An ideal P of A is called a minimal prime ideal of A:
$1) P \in Spec(A);\quad 2)\ If\ there\ exists\ Q \in Spec(A)\ such\ that\ Q \subseteq P,\ then\ P = Q.$
We denote by $Min(A)$ the set of all minimal prime ideals of A.

Theorem 1.4. *[4] Let S be a $\wedge-$closed system of A and $I \in Id(A)$ such that $I \cap S = \emptyset$. Then there exists a prime ideal P of A such that $I \subseteq P$ and $P \cap S = \emptyset$.*

Definition 1.5. [4] Let X be a nonempty subset of A.
• $Ann_A(X)$ is the annihilator of X defined by $Ann_A(X) = \{a \in A : a \wedge x = 0, \forall x \in X\}$.
• The set of all zero-divisors of X is denoted by $Z_X(A)$ and is defined as $Z_X(A) = \{a \in A : \exists 0 \neq x \in X\ such\ that\ x \wedge a = 0\}$.
Zero element of an $MV-$algebra is a zero-divisor, which is called trivial zero divisor. We denote by Z_A the set of all zero divisors of A. [3]

Lemma 1.6. *[3] If $0 \neq x \in A$, then there exists $P \in Min(A)$ such that $x \notin P$.*

Theorem 1.7. *[3] Let $P \in Min(A)$ and I be finitely generated ideal. Then $I \subseteq P$ if and only if $Ann(I) \nsubseteq P$.*

Note: Let $a \in A$. Define $P_a = \bigcap\{P : P \in Min(A), a \in P\}$.

2 Some results in zero-divisors

Theorem 2.1. *For all $x \in Z_A$, there exists $P \in Spec(A)$ such that $x \in P$ and $P \subseteq Z_A$.*

Proof. Obviously, Z_A^c is a $\wedge-$closed system and $I_0 = \{0\} \cap Z_A^c = \emptyset$, so by Theorem 1.4, there exists a prime ideal P_0 of A such that $I_0 \subseteq P_0$ and $P_0 \cap Z_A^c = \emptyset$. So $P_0 \subseteq Z_A$. Put $I_x := (x]$, for all $x \in Z_A$. We show that $I_x \cap Z_A^c = \emptyset$. Let $y \in I_x \cap Z_A^c$. Then $r \wedge y \neq 0$, for all $0 \neq r \in A$, and there exists $n \in \mathbb{N}$ such that $y \leq nx$. Since $x \in Z_A$, there exists $0 \neq r' \in A$ such that $r' \wedge x = 0$. Then $r' \wedge y \leq r' \wedge nx \leq n(r' \wedge x) = 0$, we have $r' \wedge y = 0$ and $y \in Z_A^c$, hence $r' = 0$ which is a contradiction. So $I_x \cap Z_A^c = \emptyset$, for all $x \in Z_A$. Therefore by Theorem 1.4, there exists a prime ideal P_x, for all $x \in Z_A$ such that $I_x \subseteq P_x$ and $P_x \cap Z_A^c = \emptyset$. We have $P_x \subseteq Z_A$, for all $x \in Z_A$. □

Corollary 2.2. *The set of all zero-divisors A contains at least one prime ideal of A.*

Theorem 2.3. *If P is a minimal prime ideal of A, then there exist $y \in A \setminus P$ and $k \in \mathbb{N}$ such that $y \wedge (kx) = 0$, for all $x \in P$.*

Proof. Obviously, $S = \{y \wedge (kx) | k \in \mathbb{N}, y \in A \setminus P\}$ is a $\wedge-$closed system of A. If $0 \notin S$, then by Theorem 1.4 and $I_0 = \{0\}$ there exists a prime ideal Q such that $Q \cap S = \emptyset$. It is a cliamed that $Q \subseteq P$ and $x \notin Q$. Let $a \in Q$ and $a \notin P$. Then $a \wedge kx \in S$. On the other hand $a \wedge kx \leq a$, hence $a \wedge kx \in Q$, which is a cotradiction. Put $k = 1$, since $1 \in A \setminus P$ then $1 \wedge x = x \in S$ thus $x \notin Q$. Which is a contradiction.

□

Corollary 2.4. *Every element of a minimal prime ideal of A is a zero divisor.*

In the following, we give an example to show that an ideal consisting entirely of zero divisors is not necessary a minimal prime ideal.

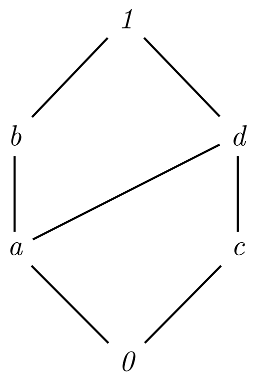
Example 2.5. Let $A = \{0, a, b, c, d, 1\}$. where $0 < a, c < d < 1$ and $0 < a < b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	b	b	d	1	1
b	b	b	b	1	1	1
c	c	d	1	c	d	1
d	d	1	1	d	1	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [4]. Obviously, $I = \{0\}$ is an ideal such that $I \subseteq Z_A$ but I is not a minimal prime ideal.



Theorem 2.6. *The set of all zero-divisors of A is a union of all minimal prime ideals of A.*

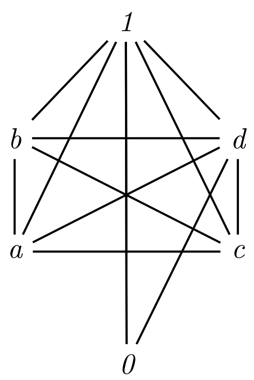
Proof. Let $x \in \bigcup_{P \in Min(A)} P$. Hence there exists $P \in Min(A)$ such that $x \in P$. We show that $x \in Z_A$. By Theorem 2.3, there exist $y \in A \setminus P$ and $k \in \mathbb{N}$ such that $y \wedge (kx) = 0$, for all $x \in P$. Since $y \wedge x \leq y \wedge (kx) = 0$, hence $x \wedge y = 0$, so $x \in Z_A$. Then $\bigcup_{P \in Min(A)} P \subseteq Z_A$. Let $x \in Z_A$. Hence there exists $0 \neq y \in A$ such that $x \wedge y = 0$. By Lemma 1.6, there exists $P \in Min(A)$ such that $y \notin P$, so $x \in P$ implies that $Z_A \subseteq \bigcup_{P \in Min(A)} P$. Therefore $Z_A = \bigcup_{P \in Min(A)} P$. □

Corollary 2.7. *The set of all zero-divisors of A is a union of prime ideals of A.*

3 Graph

Definition 3.1. χ is called a zero divisor graph if the elements of A are vertices and for any $x, y \in A$, an edge xy which is denoted by $x - y$ exists iff $Z_x \cap Z_y = \{0\}$.

Example 3.2. According to Example 2.5, we have



Theorem 3.3. *Zero divisor graph is connected.*

Proof. Obviously, $Z_1 \cap Z_x = \{0\}$, for all $x \in A$. So there is an edge between 1 and all other elements of A. Hence, by a path we can connect every two vertices, thus zero divisor graph is connected. □

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On the A_α-energy and A_α-resolvent energy of graphs

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Abstract

Let G be a graph of order n. Also, let $A(G)$, and $D(G)$ be the adjacency matrix and the diagonal matrix of the degrees of G, respectively. Nikiforov defined the matrix $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, where $0 \le \alpha \le 1$. The resolvent energy of a graph G is defined as $ER(G) = \sum_{i=0}^{n} \frac{1}{n-\lambda_i}$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the adjacency matrix of G. In this paper we calculate energy and resolvent energy of $A_\alpha(G)$ for some graph G.

Keywords: Spectrum of graph, graph energy, resolvent energy, adjacency matrix.

Mathematics Subject Classification [2010]: 05C50

1 Introduction

The energy of a graph is a quantity based on spectrum of graph. Nowadays, it is so interesting topic between researchers who study spectral graph theory. The energy of a graph, defined as the sum of its absolute eigenvalues, was first defined by Ivan Gutman in 1978 [1].

There are many kinds of graph energies, one of the more recent of these types of graph energy is resolvent energy. In [2], Gutman et al. introduced the resolvent energy of undirected graphs. Let G be a graph and $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of the adjacency matrix of G, then

$$ER(G) = \sum_{i=0}^{n} \frac{1}{n-\lambda_i}.$$

Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of the degrees of G, respectively. Nikiforov, in [3], defined the matrix $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, where $0 \le \alpha \le 1$. Obviously,

$$A_0(G) = A(G), \quad A_1(G) = D(G).$$

In this paper, we calculate the energy and resolvent energy of $A_\alpha(G)$ for some graph G. Throughout this paper, graphs are assumed to be finite, undirected and simple.

[1]speaker

2 Main Results

Theorem 2.1. *[3] Let A and B be Hermitian matrices of order n; and let $1 \leq i, j \leq n$. Then*
1) $\lambda_i(A) + \lambda_j(B) \leq \lambda_{i+j-n}(A+B)$ *if* $i + j \geq n+1$,
2) $\lambda_i(A) + \lambda_j(B) \geq \lambda_{i+j-1}(A+B)$ *if* $i + j \leq n+1$.
In either of these inequalities equality holds if and only if there exists a nonzero n-vector that is an eigenvector to each of the three eigenvalues involved.

As a result of above theorem, we have:

Corollary 2.2. *Let A and B be Hermitian matrices of order n; and let $1 \leq k \leq n$. Then*

$$\lambda_k(A) + \lambda_{min}(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda_{max}(B).$$

Theorem 2.3. *Let $0 \leq \beta < \alpha \leq 1$. If G is a graph of order n with $A_\alpha(G) = A_\alpha$ and $A_\beta(G) = A_\beta$, then*

$$\lambda_k(A_\alpha) - \lambda_k(A_\beta) \geq 0$$

for any $1 \leq k \leq n$. If G is connected, then above inequality is strict, unless $k = 1$ and G is regular.

Theorem 2.4. *If $0 \leq \beta < \alpha \leq 1$, then $ER(A_\beta) \leq ER(A_\alpha)$.*

In the sequel, let G be a graph of order n and $V(G) = \{1, 2, \ldots, n\}$. Consider $A(G) = A$ to be its adjacency matrix and $A_\alpha(G) = A_\alpha$ to be the convex linear combinations of $A(G)$ and $D(G)$, which is defined above.

Theorem 2.5. *Let δ and Δ be the minimum and maximum degree of graph G, respectively. Then*

$$\alpha\delta + (1-\alpha)\lambda_k(A) \leq \lambda_k(A_\alpha) \leq \alpha\Delta + (1-\alpha)\lambda_k(A),$$

for $1 \leq k \leq n$.

Theorem 2.6. *Let $d(1) \geq d(2) \geq \ldots \geq d(n)$ be the degree sequence of graph G. Then, for $1 \leq k \leq n$.*

$$E(A_\alpha) \leq \Sigma_{k=1}^{n} d(k)$$
$$ER(A_\alpha) \leq \Sigma_{k=1}^{n} \frac{1}{n - d(k)}.$$

Theorem 2.7. *Let $G = C_n$, the cycle with n vertices. Then*

$$0 \leq E(A_\alpha) \leq 2n, \quad \frac{n}{n - 4\alpha + 2} < ER(A_\alpha) < \frac{n}{n-2}.$$

Theorem 2.8. *Let $G_1 = K_n$ and $G_2 = K_{a,b}$ be the complete of order n and complete bipartite graph. Then*

$$E(A_\alpha(G_1)) = (n-1)(1 + |\alpha n - 1|), \quad ER(A_\alpha(G_1)) = \frac{n(2-\alpha)}{n(1-\alpha)+1}.$$

and

$$E(A_\alpha(G_2)) = \begin{cases} \alpha(2ab - a - b) + \alpha(a+b) & \alpha \geq 1/2 \\ \alpha(2ab - a - b) + \alpha(a+b) + \sqrt{\alpha^2(a+b)^2 + 4ab(1-2\alpha)} & \alpha < 1/2 \end{cases}$$

$$ER(A_\alpha(G_2)) = \frac{b-1}{n - \alpha a} + \frac{a-1}{n - \alpha b} + \frac{2n - \alpha(a+b)}{n^2 - n\alpha(a+b) - ab(1-2\alpha)}.$$

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Distance Matrices and Spectral Radii of Graphs

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Abstract

Distance matrix of a graph is a symmetric matrix which encodes distances between all pairs of vertices in the graph. In 1971, Graham and Pollack showed that the so-called addressing problem in data communication systems is related to the number of negative eigenvalues of the distance matrix. They also showed that the determinant of the distance matrix of a tree depends only on the number of its vertices. Since then, this topic has been an active area of research, and its combinatorial and spectral aspects have been extensively studied. After giving a brief overview of the subject, I will discuss some recent extremal problems regarding the distance spectral radius, that is, the largest eigenvalue of the distance matrix. I will also present our recent extremal results on the distance spectral radius of bicyclic graphs.

Keywords: Distance Matrix, Graph Theory, Distance Spectral Radius, Byciclic Graphs

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Investigating and analyzing the load balance produced in the Internet of Things and the solution to reduce it by cloud computing

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Abstract

The Internet of Things is increasing the presence of computing devices across the entire network in public, business and private spaces. These devices do not simply act as sensors, but play a major role in computing, storage, and network resources. Cloud computing plays a very important role in processing large amounts of data. With the advent of the Internet of Things, large amounts of data are generated from these devices. Therefore, we need a cloud feature to be close to the request generator, so that the processing of this huge amount of information is one step closer to the end user. This led to the emergence of cloud computing to provide storage and computing at the edge of the network, which reduces network traffic and fixes many computer problems. The Internet of Things was placed on the edge of the network so that applications could use distributed resources . This concept is known as fog Computing. Computations bring resources closer to end users by calculating and categorizing However, a new computational paradigm is emerging and still needs to be standardized. .

Keywords: Fog, Fog Computing, Internet of Things, Load Balancing
Mathematics Subject Classification [2010]: Primary: 22D15, 43A10; Secondary: 43A20, 46H25

1 introduction

In the last few years, major advances in information and communication technology have been evident. Since most of Internet traffic comes from or ends up in the cloud [1, 2, 3], it is estimated that approximately two-thirds of the total work done by Traditional IT services such as data collection and processing are processed on the cloud. Enable cloud computing for users Cloud computing is a kind of integration and development of network computing, distributed computing and parallel computing. the tendency to use clouds has been adapted to improve its efficiency, which reduces CAPEX and OPEX capabilities, and improves the flexibility and scalability of the entire network.with the advent of internet, a large number of smart devices and objects are connected to the internet today and produce a large amount of information every second. Cisco estimates that around 50 billion devices, namely the object, are connected to the Internet networks by 2022. Although the cloud can provide storage and processing of efficient data, the increasing volume of data causes an increase in energy consumption, a heavy load on the communication bandwidth and "unacceptable" delay. Furthermore, since the cloud is relatively distant since the use of the Internet devices, the importance and novelty of it may be lost when the data are to be stored for storage or processing [4].To meet cloud constraints due to the time of Internet delay, Cisco has proposed the concept of computing in 2012 to address the challenges of Internet applications.The OpenFog consortium, including Cisco, Dell, Intel, and Microsoft implemented it in 2015. The security of mobile edges and Fog is one of

[1]speaker

the major challenges for successful implementation and deployment of Internet infrastructure [1, 5, 6, 7]. The Fog Computing reflect the structural basis of the Internet device, where many of the functionality and coordination of network devices exist. Cell cells, for example, are implemented instead of involving the cloud in order to process data close to data sources and lost data [5, 8]. This reduces communication delays and better utilization of available computing, storage and better utilization of network resources in Fog.

2 Fog Computing

Typically, the Fog computation consists of the elements of the traditional grid. For example, routers, switches, upper boxes, proxy servers, base stations, etc. can be placed near the Internet / IP devices. Despite the very good benefits of Esme " sComputing , some worry it. This involves the way of the interaction of the Cloud - 2 - Fog (C2F)and the Fog - 2 - Fog (F2F)co - operation[9]. The article [9] addresses the issue of F2F by promoting service service among the May nodes, at least, the time delay for the Internet services. Dastjerdi et al. Have defined fog computing as computation distribution of service expansion range from cloud computing one step away from the user[10]. Based on the large - scale and nature of the paper [9], it discusses the study of load balancing based on genetic algorithm and cloud computing platform. this paper examines a two - performance evaluation algorithm based on improved genetic algorithm and looks at the average time time of users due to the tasks sent to improve the overall satisfaction.moreover, according to a comprehensive virtual machine load index, this paper examines the development of load balancing algorithm based on genetic algorithm and proves that this method can meet the load balancing requirements under the cloud environment and improve the ratio of utilization and load balancing of resources. The calculation of the interface between cloud computing and the end machines may be considered as building blocks. Based on the article [11], the features of the Fog may be summarized as follows:
1) Knowing the location and reducing the time delay 2) Geographic distribution 3) scalability. 4) Support of mobility 5) The real - time interactions 6) heterogeneity 7) Interoperability

3 Research literature

A lot of work has been done to balance the load in order to improve performance and prevent excessive use. Various load balancing algorithms include the robin round (RR, Min - Min, Max - Min, and so on. load balancing algorithms are divided into two main categories namely static and dynamic. Several algorithms have been proposed for load balancing on the cloud and Fog. The algorithm manages the client's request and schedule its duties. load balancing algorithms for each virtual machine are allocated for the efficiency of all resources and avoiding the imbalance between the load in virtual machines. the common algorithms used to allocate resources (RR, (Job and Shortest Job (SJF)and Artificial Bee (GA)and Genetic Algoritm (GA)and In [12], the authors examined charge related to charging and discharging of electric vehicles. The interaction between EVs and EV is defined in supply Station (EVPSS)and cloud processing. The timing or schedule is based on the use of the resource allocation algorithm based on the priority and non - Calender of the resource allocation algorithm.

The Fog processing platform is used to manage the machines ' calculations in [13] to raise the response rate by lowering the delay. In [1] the authors considered the energy management system (EMS)for the mobile. Mobile phones always retrieve data from the cloud. however, the retrieval of information from the cloud is remote and this time will prolong the response. therefore, the processing concept at the edge of time improves the response and the response is recovered faster.[14] has been used to allocate data Center based data based on minimum response time from pso and Simulated annealing and service interface policy. Shortest Job First (SJF) algorithm in [15]. Provided for load restraint. The algorithm is executed in two different scenarios. In the first scenario, one data center is assigned to twenty-five virtual machines and in the second scenario, two data centers are assigned to fifty virtual machines. The second scenario shows better results than the first scenario. The proposed technique is also compared with the RR and Esce methods and the SJF is evaluated for better results in the proposed scenario.

The authors in [16] presented real - time dynamic value model and time application for users in real - time environments. It is considered the charge of charging and discharging the charge of electric vehicles to reduce the load peak and effective management of the request. DFID technology is a decentralized cloud processing architecture. The real time from Toronto is used for simulation. The authors [17] used the RR, Throttle, and PSO to balance load balance for the virtual machine. two clustering methods have been proposed for experimental experiments. Each cluster is individually linked by a fog. The overall performance of the PSO is better than RR. Van et al., in [18], proposes a load - based load balancing and energy - based scheduling method by focusing on the complex problems of energy consumption in production clusters. First, an energy consumption model related to workload is created on the node, and an optimization function is formulated with the objective of load balancing of the production cluster. the improved particle swarm optimization algorithm is then used to obtain an optimal solution and the priority for achieving tasks is made to the production cluster. Finally, a multi-agent system is introduced to achieve the distributed scheduling of the production cluster. the proposed method has been validated through experiments with sweet packaging thread and experimental results showed that the proposed method provides optimum planning and load balancing for collaborative robots. In this paper, [19] address the challenging issue of multiple users requiring unloading, in which all requests must be processed by local computing resources. They proposed a set of small complex clusters with low complexity and a customizable resource management algorithm for fog clustering. The simulation results show that the proposed algorithm achieves a high user satisfaction rate of at least 90 percent for 4 users per small cell, medium power consumption, or high gain. In [20] suggests a new load - balancing technique for the validation of the candidate and finds a lesser time assignment for task allocation. The proposed load - balancing technique is more efficient than other existing approaches in finding edcs times less for task allocation. The proposed approach not only improves the efficiency of load - load efficiency; it strengthens security by verifying the validity of the destination Ningning et al. [21] investigate the Fog framework and use atomization technology to convert physical nodes at different levels to virtual machine nodes. Accordingly, this paper uses the graph partitioning theory to create a computational load balancing algorithm based on dynamic graph partitioning. The simulation results show that a cloud computing framework can create flexible system network flexibility after cloud computing, and the dynamic load balancing mechanism can efficiently configure system resources as well as reducing the consumption of node transmission by system changes efficiently

4 Conclution

In recent years, the Internet of Things has attracted the attention of most researchers. There were many challenges to it and one of the challenges was the use of the cloud, which led to the concept of fog. Fog Computing also have their challenges and need to be standardized. One of the important issues in fog balance calculations that should be paid attention to and in this article we have examined several dynamic and static load balancing algorithms.

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Coherent Closure of Some Classes of Graphs

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Abstract

In this paper, we study the coherent closure of graphs. We determine the coherent closure of a disconnected graph by using the coherent closure of its components. Moreover, we determine the coherent closure of cocktail party graphs and also complete multipartite graphs, and show that we can describe them in the terms of wreath product and direct sum of trivial and discrete coherent configurations.

Keywords: coherent configuration, coherent closure, graph
Mathematics Subject Classification [2010]: Primary: 05E30, 20E22; Secondary: 05C25

1 Introduction

The theory of coherent configuration is related to graphs, matrix algebras and groups. In graph theory coherent configuration is used for studying the Graph Isomorphism Problem.

In [10], B. Weisfeiler and A. Leman have shown that a special matrix algebra is assigned to a given graph which contains the adjacency matrix of the graph. In fact, this algebra is the adjacency algebra of a coherent configuration. The *coherent closure of a graph* is the smallest coherent configuration on the vertex set of the graph such that the edge set of which is the union of some basic relations of the coherent configuration.

The coherent configuration of some classes of graphs, which are called algebraic forests, have been studied in [4]. And the class of *forestal coherent configurations* have been defined inductively by means of direct sums and wreath products in [4]. Also, in [5, 9, 6, 8] by using coherent configuration Weisfeiler-Leman Algorithm for some classes of graphs have been studied.

This paper is organized as follows. In the rest of this section, we present some basic definitions and notations on coherent configurations. In Section 2, we first remind the concept of coherent closure of a graph. Then, we give our main results.

1.1 Coherent configuration

Let Ω be a non-empty finite set of order n. Define 1_Ω as follows, $1_\Omega = \{(\alpha, \alpha) \in \Omega^2 : \alpha \in \Omega\}$. Let S be a set of non-empty binary relations on Ω such that partitions Ω^2. The set of all unions of the relations of S is denoted by $S^\cup$. The set S^*, is defined to be the set of all $s^*, s \in S$, where $s^* = \{(\beta, \alpha) : (\alpha, \beta) \in s\}$. For each $\alpha \in \Omega$ and $s \in S$ the neighborhood of α in s is defined by, $\alpha s = \{\beta \in \Omega : (\alpha, \beta) \in s\}$.

Definition 1.1. The pair $\mathcal{X} = (\Omega, S)$ is called a *coherent configuration* if it satisfies the following conditions:

1) $1_\Omega \in S^\cup$;

2) $S^* = S$;

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3) For any $r, s, t \in S$ there exists an *intersection number* $c_{r,s}^t$ such that $c_{r,s}^t = |\alpha r \cap \beta s^*|$ does not depend on the choice of $(\alpha, \beta) \in t$.

The elements of Ω and S are called *points* and *basis relations* of $\mathcal{X}$, respectively. The numbers $|\Omega|$ and $|S|$ are called the *degree* and the *rank* of $\mathcal{X}$, and are denoted by $\deg(\mathcal{X})$ and $\mathrm{rk}(\mathcal{X})$, respectively. Two examples of coherent configuration are the *trivial* and *discrete* coherent configuration on n points, $\mathcal{T}_n$ (or $\mathcal{T}_\Omega$) and $\mathcal{D}_n$ (or $\mathcal{D}_\Omega$), respectively, where the trivial one contains 1_Ω and its complement in Ω^2 (if $n \geq 2$). The discrete one contains all n^2 singleton relations $\{(\alpha, \beta)\}$, with $\alpha, \beta \in \Omega$.

There is a natural partial order $\leq$ on the set of all coherent configurations on Ω with the smallest and greatest one equal to $\mathcal{T}_n$ and $\mathcal{D}_n$, respectively. For coherent configurations $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega, S')$ we define,

$$\mathcal{X} \leq \mathcal{X}' \Longleftrightarrow S^{\cup} \subseteq (S')^{\cup}.$$

The wreath product and direct sum of coherent configurations $\mathcal{X}$ and $\mathcal{X}'$ are denoted by $\mathcal{X} \wr \mathcal{X}'$ and $\mathcal{X} \boxplus \mathcal{X}'$, respectively.

If there exists a bijection between the point sets of two coherent configurations that induces a bijection between their sets of basis relations, then these two coherent configurations are *strong isomorphic*, or shortly *isomorphic*.

2 Main Results

By a graph we mean a finite simple an undirected graph, $X = (\Omega, E)$, where Ω is the vertex set and $E \subseteq \Omega \times \Omega$ is the edge set of X. The complement of the graph X, is denoted by $\bar{X}$.

An isomorphism of graphs $X = (\Omega, E)$ and $X' = (\Omega', E')$ is a bijection $f : \Omega \to \Omega'$ such that any two vertices u and v of X are adjacent in X if and only if $f(u)$ and $f(v)$ are adjacent in X'. If an isomorphism exists between two graphs, then the graphs are called isomorphic.

The cocktail party graph on $2n$ vertices, is the graph consisting of two rows of paired vertices in which all vertices but the paired ones are connected with an edge of the graph. A k-partite graph is a graph whose vertices are or can be partitioned into k different independent sets. A complete k-partite graph is a k-partite graph in which there is an edge between every pair of vertices from different independent sets, denoted by $K_{n_1,\ldots,n_k}$, where the number of vertices in the i-th independent set is n_i for $1 \leq i \leq k$.

2.1 Coherent closure of a graph

We can consider each graph as a union of basis relations of a coherent configuration, for an example consider discrete coherent configuration. Now, between all of such coherent configuration, the smallest one which is also the intersection of them, is called the coherent configuration generated by the graph. This coherent configuration is exactly generated by Weisfeiler-Leman algorithm. For a graph X, this coherent configuration named coherent closure of X.

Definition 2.1. Let $X = (\Omega, E)$ be a graph, $|\Omega| = n$. We define $\mathfrak{T}$ and $WL(X)$ as follow:

$$\mathfrak{T}(X) = \{\mathcal{X} = (\Omega, S) : \mathcal{X} \leq \mathcal{D}_n, \quad E \subseteq S^{\cup}\}, \qquad WL(X) = \bigcap_{\mathcal{X} \in \mathfrak{T}(X)} \mathcal{X}.$$

Then, the coherent configuration $WL(X)$ is called the *coherent closure* of the graph X.

First we describe the coherent closure of some disconnected graphs. In the next two theorems we show that they are constructed by wreath product and direct sum of coherent configurations.

Theorem 2.2. *Let $X_1, X_2, ..., X_m$ be graphs, such that all of them are isomorphic to each other. Also let X be the disjoint union of X_i, $1 \leq i \leq m$. Then $WL(X)$ is isomorphic to $WL(X_1) \wr \mathcal{T}_m$.*

Proof. Since for each i, j, $1 \leq i, j \leq m$ and $i \neq j$, X_i and X_j are isomorphic, so there is a strong isomorphism from $WL(X_i)$ to $WL(X_j)$. Moreover, each strong isomorphism is a weak isomorphism, thus using Theorem 5.1 of [4] completes the proof. □

Theorem 2.3. *Let $X_1, X_2, ..., X_m$ be graphs such that for each i, j, $1 \leq i, j \leq m$ and $i \neq j$, X_i and X_j are not isomorphic. Also suppose that X be the disjoint union of X_i, $1 \leq i \leq m$. Then $WL(X)$ is isomorphic to $\boxplus_{i=1}^{m} WL(X_i)$.*

By Theorem 2.2 and Theorem 2.3, we can determine the coherent closure of each disconnected graph by using the coherent closure of its components.

In the next proposition we show the relation between coherent closure of a graph and its complement.

Proposition 2.4. *Let X be a graph. Then: $WL(X)$ is isomorphic to $WL(\bar{X})$.*

Proof. Let $WL(X) = (\Omega, S)$. Since $E(\bar{X}) = \Omega \times \Omega \setminus \{1_\Omega, E(X)\}$ and $E(X)$ is a union of some basis relations of S. Thus $E(\bar{X})$ is also a union of some basis relations of S too. Therefore, $WL(X)$ is also the coherent closure of $\bar{X}$. □

If we cosider the complete graph on n points, then its is easy to show that its coherent closure is the trivial coherent configuration on n points. The coherent clouser of distance regular graphs, especially strongly regular graphs and also cycles, have been studied in the following papers [2, 3, 11]. Now, we want to specify the coherent closures of cocktail party graphs and also complete multipartite graphs.

The following proposition provides a necessary and sufficient condition for a coherent configuration to be the coherent closure of a cocktail party graph or its complement.

Proposition 2.5. *Let X be a graph. Then, $WL(X)$ is isomorphic to $\mathcal{T}_2 \wr \mathcal{T}_n$ if and only if X is a cocktail party graph on $2n$ points or its complement.*

In the following proposition we determine the coherent closure of a complete multipartite graph.

Proposition 2.6. *Let X be a graph. Then,*

1) *X is a complete k-partite graph $K_{\underbrace{n, ..., n}_{k-times}}$ or its complement if and only if $WL(X)$ is isomorphic to $\mathcal{T}_n \wr \mathcal{T}_k$;*
2) *if X is a complete k-partite graph $K_{n_1,...,n_k}$, such that for each i, j, $i \neq j$, $n_i \neq n_j$, then $WL(X)$ is isomorphic to $\boxplus_{i=1}^{k} \mathcal{T}_{n_i}$.*

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On the maximum cardinality of the number of pairwise non-adjacent vertices and edges of random subgraphs

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Abstract

In this paper we will verify the matching of random spanning subgraphs of the n-cube with asymptomatic probability tending to 1.

Keywords: Random subgraphs, Bipartite graph, Matching set, Markov's inequality.
Mathematics Subject Classification [2010]: Primary 05C80

1 Introduction

The n-cube Q_n is the graph consisting of the 2^n vertices $(a_1, a_2, ..., a_n)$, $a_l \in \{0, 1\}$, and the $n2^{n-1}$ edges between vertices are distinct in exactly one coordinate. A spanning subgraph g of Q_n has the same vertex set as Q_n. An spanning subgraph f of Q_n with the vertex set $A \subseteq Q_n$ contains exactly those edges of Q_n that join two vertices in A. Choosing independently the edges of g (the vertices of f) at random, with the same probability p, at a random spanning subgraph with following probabilities so we have $\text{Prob}(g) = p^{|\theta|}q^{n2^{n-1}-|\theta|}$ and $\text{Prob}(f) = p^{|f|}q^{2n-|f|}$, respectively, where $q = 1 - p$.

For example, for fixed $p > 1/2(p < 1/2)f$ is a.s. (not) connected, where a.s. means almost surely [5]. More precisely, the number of isolated vertices is asymptotically Poiss(1/2) that is particularly Prob(f is connected)$\sim$ Prob(f has no isolated vertices) $\sim 1/\sqrt{e}$ ([4]). For random spanning subgraphs, we replace only $\lambda = 1/2$ by $\lambda = 1$ and so $1/\sqrt{e}$ by $1/e$ ([1, 2, 6]).

2 Matching set of random spanning subgraphs

Let (X_n), $n = 1, 2, ..,$ be a sequence of non-negative integer-valued random variables. Then by Markov's inequality

$$\text{Prob}(X_n < \varphi_n EX_n) \to 1 \tag{1}$$

as $n \to \infty$, where EX_n is the expectation of X_n and $\varphi_n \to \infty$ arbitrarily slowly as $n \to \infty$[3]. In particular, (1) implies that

$$\text{Prob}(X_n = 0) \to 1 \text{ if } EX_n \to 0. \tag{2}$$

If there are sequences $\varepsilon_n \to \infty$ and α_n such that $\text{Prob}(|X_n - \alpha_n|) < \varepsilon_n\alpha_n$ we will say "X_n is a.s. asymptotically α_n" and use the short notation "$X_n \sim \alpha_n$ a.s.". A set of pairwise non-adjacent vertices of a graph G is called a matching (vertex) set. The maximum cardinality of a matching set of G is said to be the matching number $\beta_0(G)$. Analogously the edge matching number $\beta_1(G)$ is the maximum number of pairwise non-adjacent edges of G. It is easy to show that

$$\beta_0(G) + \beta_1(G) = |V| \tag{3}$$

for every bipartite graph $G = (V, E)$.

[1]speaker

Theorem 2.1. *If* $pn \to \infty$ *then*

$$\beta_0(g) \sim 2^{n-1} a.s.$$

and

$$\beta_1(f) \sim p2^{n-1} a.s.$$

Proposition 2.2. *If* $p > 1/2$ *fixed then a.s.*

$$\beta_0(g) = 2^{n-1}$$

and

$$\beta_0(f) = |f|/2$$

provided that If $|f|$ *is even.*

For $p = 1/2$ there exist isolated vertices with positive probability. Also for p with $pn \to 0$ asymptotically all vertices both of f and g are isolated so that for these probabilities $\beta_0(f) \sim p2^n$ and $\beta_0(g) \sim 2^n$ a.s. We do not know β_0 if p has the order of magnitude $1/n$.

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